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ONE OR TWO FREQUENCIES? THE SYNCHROSQUEEZING ANSWERS

HAU-TIENG WU*,‡, PATRICK FLANDRIN†,§
and INGRID DAUBECHIES*,¶

*Mathematics Department, Princeton University,
Washington Road, Princeton, NJ 08544-1000, USA

†Laboratoire de Physique (UMR 5672 CNRS),
Ecole Normale Supérieure de Lyon,
46 allee d'Italie, 69364 Lyon Cedex 07, France

‡hauwu@math.princeton.edu
§patrick.flandrin@ens-lyon.fr
¶ingrid@math.princeton.edu

The synchrosqueezed transform was proposed recently in [Daubechies *et al.* (2009)] as an alternative to the empirical mode decomposition (EMD) [Huang *et al.* (1998)], to decompose composite signals into a sum of "modes" that each have well-defined instantaneous frequencies. This paper presents, for synchrosqueezing, a study similar to that in [Rilling and Flandrin (2008)] for EMD, of how two signals with close frequencies are recognized and represented as such.

Keywords: EMD; synchrosqueezing; time-frequency analysis; beating.

1. Introduction

The empirical mode decomposition (EMD) algorithm, first proposed in [Huang *et al.* (1998)], made more robust as well as more versatile in [Huang *et al.* (2009)], is a technique that aims to decompose functions into their building blocks, when the functions are the superposition of a (reasonably) small number of components. The components are assumed to be well separated in the time-frequency plane, and all of them are with slowly varying amplitudes and frequencies. The EMD has already shown its usefulness in a wide range of applications including meteorology, structural stability analysis, and medical studies — see [Huang and Wu (2008)]. On the other hand, the EMD algorithm contains heuristic and ad hoc elements that make it hard to analyze mathematically.

‡Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA.

The paper by [Daubechies *et al.* (2009)] proposed and analyzed a method called *synchrosqueezing* that captures the flavor and philosophy of the EMD approach, albeit using a different approach in constructing the components. A precise mathematical definition for a class of functions that can be viewed as a superposition of a reasonably small number of approximately harmonic components was given, and for functions in this class, it was proved that the method succeeds indeed in decomposing arbitrary functions belonging to this class.

Synchrosqueezing by definition is a highly nonlinear operator the behavior of which is sufficiently complicated to generate some interesting phenomena. In this paper we demonstrate such an interesting phenomenon, also observed for EMD. When a signal is composed of two components with close instantaneous frequencies, EMD exhibits a beating phenomenon [Rilling and Flandrin (2008)]. More precisely, if a signal is composed of two harmonics, i.e., $f(t) = \cos(2\pi t) + a\cos(2\pi\xi_0 t)$, Rilling and Flandrin [2008] show an interesting zone in the a vs. ξ plane (amplitude vs. frequency plane) where EMD misidentifies the sum of two components as only a single component; the precise shape of this zone depends on the value ξ_0. They also quantified this phenomenon carefully and called it *beating*, because EMD "identified" the two harmonics as a single oscillating (i.e., beating) signal. Here, we study this phenomenon for the EMD-inspired synchrosqueezing. We provide numerical results as well as an analysis of these results.

2. Synchrosqueezing

Synchrosqueezing was originally introduced in the context of analyzing auditory signals [Daubechies and Maes (1996)]; in [Daubechies *et al.* (2009)] it was shown to catch the flavors of EMD [Daubechies *et al.* (2009)]. It is in fact a special case of reassignment methods [Auger and Flandrin (1995); Chassande–Mottin *et al.* (2003, 1997)]. In synchrosqueezing, one reallocates the coefficients resulting from a continuous wavelet transform based on the frequency information, to get a concentrated picture over the time-frequency plane, from which the instantaneous frequencies are then extracted. Special properties of synchrosqueezing include that (1) it is adaptive to the given signal; (2) the signal can be reconstructed from the reallocated coefficients. We refer the readers to [Daubechies *et al.* (2009)] for the motivation, details of the algorithm, and detailed discussion. We briefly list the main steps of the algorithm here.

(1) REQUIRE: A signal $f(t)$; a mother wavelet $\psi(t)$ with $\text{supp}\hat{\psi}(\xi) \subset [1-\Delta, 1+\Delta]$, where Δ small enough;
(2) (Step 1) Calculate the continuous wavelet transform $W_f(a, b)$ of f.
(3) (Step 2) Calculate the instantaneous frequency information $\omega(a, b)$.
(4) (Step 3) Calculate the synchrosqueezed function $S_f(\xi, b)$ over the time-frequency plane.

(5) (Optional) Extract dominant curves from $S_f(\xi, b)$.
(6) (Optional) Reconstruct the signal as a sum of components, one for each extracted dominant curves.

3. Numerical Experiment

In [Rilling and Flandrin (2008)], the beating phenomenon stemming from the local extremal points detection (part of the EMD procedure) was studied and reported in detail. Since synchrosqueezing has been shown to be similar to EMD [Daubechies *et al.* (2009)], it would be interesting to study its behavior in the same situation, and to see whether it exhibits a similar beating phenomenon. For some signals, it is not immediately clear whether they should be decomposed into a sum of several simple summands, like sinusoids, or be preserved and viewed as a single modulated signal. If an oracle were whispering in our ear information about the underlying physical rule, it would be easy to decide. However, in general, no such oracle is available, and we have only the signal itself as a guide; the information we read from the signal may, however, depend on the algorithm we use. Thus, the first step toward the answer to this question is understanding how the algorithm reacts to the simple case where the signals are the composite of two harmonic functions with different frequencies.

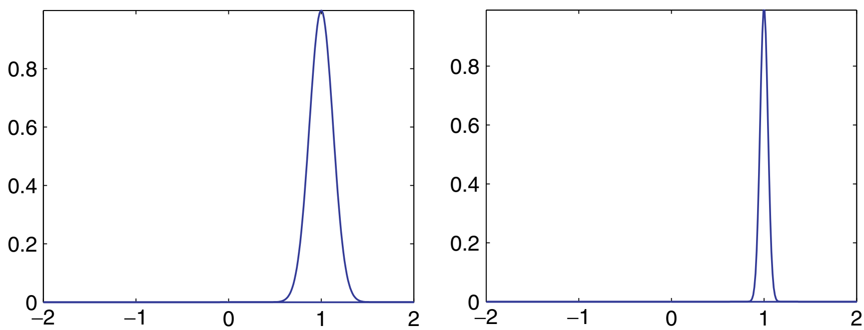
Consider two discrete time harmonic signals $f_1[n] = \cos(2\pi \frac{n}{T})$ and $f_2[n] = A\cos(2\pi\xi_0 \frac{n}{T})$ (where $1/T$ is significantly larger than the Nyquist rate – 1 in this case — to avoid any possible confounding) with $A > 0$ and $0 < \xi_0 < 1$. We run the synchrosqueezing algorithm to analyze $f[n] = f_1[n] + f_2[n]$ and ask the same questions as in [Rilling and Flandrin (2008)], namely: given $f[n] = f_1[n] + f_2[n]$, (1) "When does synchrosqueezing retrieve the two individual tones?," (2) "When does it consider the signal as a single component?," and (3) "When does it do something else?."

We introduce the following "error function" e to measure how accurately the synchrosqueezing can extract f_1

$$e(S_f, S_{f_1}) = \frac{\sum_{i,j} |[\Re S_f(\xi_i, b_j) - \Re S_{f_1}(\xi_i, b_j)]\Re S_{f_1}(\xi_i, b_j)|}{\sum_{i,j} |\Re S_{f_1}(\xi_i, b_j)|^2}, \quad (1)$$

where $\{\xi_i\}$ and $\{b_j\}$ are numerical discretization when calculating S_f and S_{f_1}. Note that the numerator is close to zero if the synchrosqueezed result S_f of the composite signal $f = f_1 + f_2$ gives the right result S_{f_1} in the area where S_{f_1} is mainly supported, i.e., when synchrosqueezing succeeds in separating the signals. By its definition, e is a function depending on A and ξ_0.

We fix T so that the sampling rate is 50 Hz, and sample for 20 seconds, $t \in [0, 20]$. Since the width of the support of $\hat{\psi}$, the Fourier transform of the mother wavelet ψ, is important in the synchrosqueezing, we test the signal based on two different mother wavelets ψ_1 and ψ_2, where $\hat{\psi}_1$ has a larger support than $\hat{\psi}_2$ (Fig. 1). The

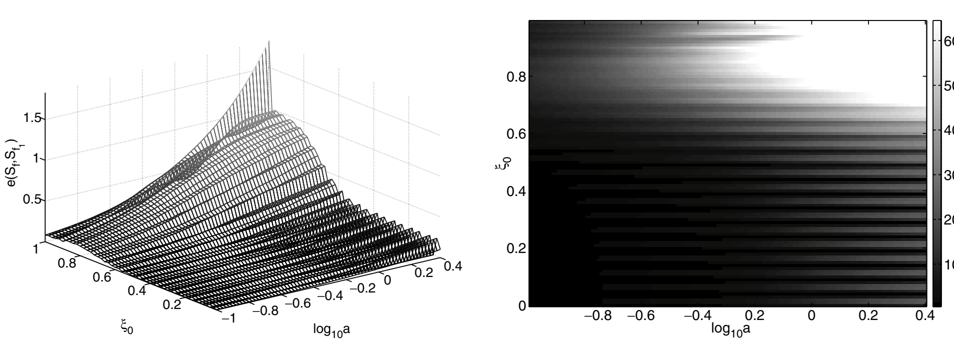


(a) (b)

Fig. 1. (a) $\hat{\psi}_1$ and (b) $\hat{\psi}_2$.

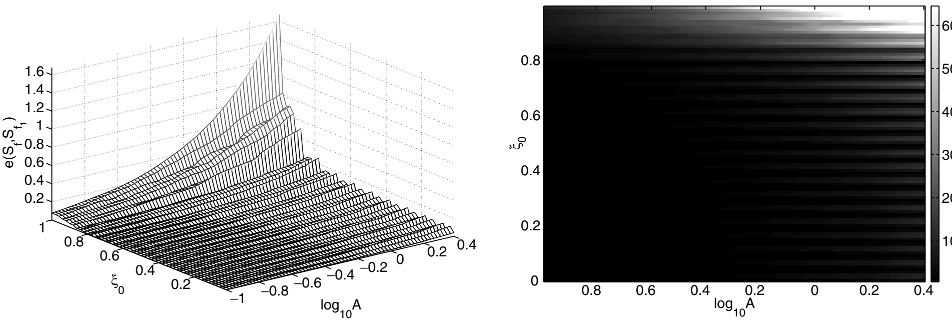
"continuous wavelet" transform is approximated by discretizing a, dividing every *octave* (i.e., an interval, in log scale, between a and $2a$) into 32 slots, equispaced in log scale. We consider $\xi_0 \in [0, 1]$ and $A \in [10^{-1}, 10^{0.4}]$. The results, using ψ_1, resp. ψ_2 as the mother wavelet, are shown in Figs. 2 and 3, respectively. The x-axis in both figures is the amplitude a, represented in log scale (from -1 to 2), and the y-axis is the frequency, represented in linear scale (from 0 to 1). The beating phenomenon shows up, as expected, when the frequency f is close to 1 and the amplitude a is large: in this case, the synchrosqueezing fails to extract two components. Moreover, the smaller the support of $\hat{\psi}$ is, the better will be the separation result. Also note that when $\xi_0 = 1$, $e = a$. That explains why we see a smoothly increasing error when $\xi_0 = 1$.

As for EMD, it appears that synchrosqueezing evidences a nonsymmetric behavior with respect to the amplitude parameter, separation becoming increasingly difficult when a is increased. However, unlike what is observed for EMD, we do not see a sharp transition from zone 1 to zone 3 when a increases, as shown in



(a) (b)

Fig. 2. Here, we use ψ_1 in the synchrosqueezing algorithm. (a) The graph of the function $e(S_f, S_{f_1})$ defined by Eq. (1). The higher the value of $e(S_f, S_{f_1})$, the worse the separation and (b) the 2D projection figure of the function $e(S_f, S_{f_1})$.



(a) (b)

Fig. 3. Here, we use ψ_1 in the synchrosqueezing algorithm. (a) The graph of the function $e(S_f, S_{f_1})$ defined by Eq. (1). The higher the value of $e(S_f, S_{f_1})$, the worse the separation and (b) the 2D projection figure of the function $e(S_f, S_{f_1})$.

Fig. 4. Theoretically, when ξ_0 is less than the threshold depending on the mother wavelet, as we derive in the next section, synchrosqueezing can distinguish the two components. The other difference is when the amplitude a is small. We see that synchrosqueezing can separate the two components more effectively than with EMD. In fact, no matter how small a is, when ξ_0 is close to 1, EMD get confused, as can be seen in the transition area from zone 1 to zone 2 in Fig. 4.

It would be interesting to understand what feature of synchrosqueezing makes the error large in these regimes. In Fig. 2, for instance, keeping the amplitude fixed at $a = 1$ and increasing ξ_0 from 0 to 1, we see that synchrosqueezing becomes increasingly "confused," since the error e increases. Figure 5 shows plots of $\log(1 + |S_f(b, \xi)|)$ for the corresponding synchrosqueezed transforms $S_f(b, \xi)$, for different values of ξ_0. When $\xi_0 = 0.9$, i.e., $f_2(t) = \cos(2\pi \times 0.9t)$, it is difficult to read off from the synchrosqueezed transform whether the signal is composed of one or two components, because the curve(s) in the dominant region keeps merging and splitting up again, as time progresses. This beating phenomenon, stemming from the nonlinearity of synchrosqueezing, is quantitatively described in the next section.

4. The Beating Phenomenon

Consider the continuous model: $f_1(t) = \cos(2\pi t)$, $f_2(t) = a\cos(2\pi\xi_0 t)$, and $f(t) = f_1(t) + f_2(t)$, where $a > 0$ and $0 < \xi_0 < 1$. Pick a wavelet $\psi \in C^\infty$ so that $\text{supp}\hat{\psi} = [1 - \Delta, 1 + \Delta]$, $\int a^{-1}\hat{\psi}(a)da = 1$ and $\xi_0 < \frac{1-\Delta}{1+\Delta}$. The continuous wavelet transforms of f_1 and f are then

$$W_{f_1}(a, b) = \sqrt{a}\hat{\psi}(a)e^{ib}$$

and

$$W_f(a, b) = \sqrt{a}[\hat{\psi}(a)e^{ib} + a\hat{\psi}(a\xi_0)e^{ib\xi_0}]$$

Over the region $Z_{f_1} = \{(a, b) \in \mathbb{R}^2 : \sqrt{a}\hat{\psi}(a)e^{ib} \neq 0\}$, by definition we have

$$\omega_{f_1}(a, b) = 1 \quad \text{for all } (a, b),$$

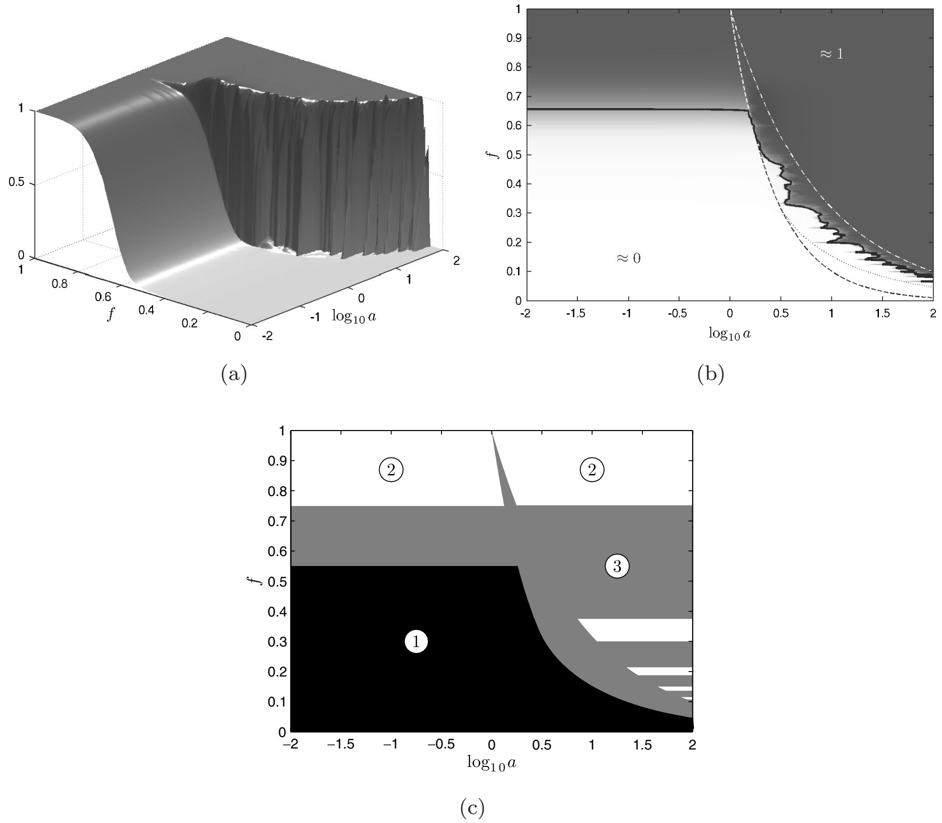


Fig. 4. Reproduction of Figs. 2 and 7 from [Rilling and Flandrin (2008)] given here to compare with Figs. 2 and 3. This figure considers the same signal, $\cos(2\pi t) + a\cos(2\pi f t)$, where $a \in [10^{-1}, 10^2]$ and $f \in (0, 1)$; note that the amplitude is denoted by the lowercase a and the frequency is denoted by the lowercase f. For details on the exact parameters used in the EMD algorithm, see [Rilling and Flandrin (2008)]. (a) The error as a function of amplitude and frequency; (b) The 2D projection onto the (a, f)-plane of amplitude and frequency; (c) Summary of [Rilling and Flandrin (2008)]. Three zones with different behaviors can be distinguished: (1) the two components are separated, (2) they are considered as a single signal, and (3) EMD does something else (© [2008] IEEE).

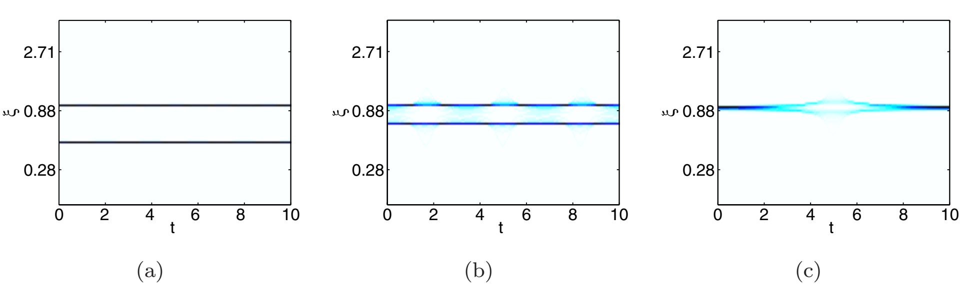


Fig. 5. Here, we use ψ_1 in the synchrosqueezing algorithm. (a) $f(t) = \cos(2\pi t) + \cos(2\pi \times 0.5t)$; (b) $f(t) = \cos(2\pi t) + \cos(2\pi \times 0.7t)$; and (c) $f(t) = \cos(2\pi t) + \cos(2\pi \times 0.9t)$.

and over the region $Z_f = \{(a,b) \in \mathbb{R}^2 : \sqrt{a}\hat{\psi}(a)e^{ib} + a\sqrt{a}\hat{\psi}(a\xi_0)e^{ib\xi_0} \neq 0\}$ we have

$$\omega_f(a,b) = \frac{\sqrt{a}\hat{\psi}(a)e^{ib} + a\xi_0\sqrt{a}\hat{\psi}(a\xi_0)e^{ib\xi_0}}{\sqrt{a}\hat{\psi}(a)e^{ib} + a\sqrt{a}\hat{\psi}(a\xi_0)e^{ib\xi_0}}.$$

Since $\xi_0 < \frac{1-\Delta}{1+\Delta}$, it follows that $a\xi_0 < 1-\Delta$ when $1-\Delta < a < 1+\Delta$. On the other hand, when $1-\Delta < a\xi_0 < 1+\Delta$ we have $a > 1+\Delta$. In conclusion, we have

$$\omega_f(a,b) = \begin{cases} 1 & \text{when } 1-\Delta < a < 1+\Delta \\ \xi_0 & \text{when } 1-\Delta < a\xi_0 < 1+\Delta \\ \text{not defined} & \text{otherwise.} \end{cases}$$

As a result,

$$\Re S_{f_1}(\xi,b) = \begin{cases} \cos(2\pi b)\delta(\xi-1) & \text{when } \xi = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Re S_f(\xi,b) = \begin{cases} \cos(2\pi b)\delta(\xi-1) & \text{when } \xi = 1 \\ a\cos(2\pi\xi_0 b)\delta(\xi-\xi_0) & \text{when } \xi = \xi_0 \\ 0 & \text{otherwise,} \end{cases}$$

which imply that after proper discretization, the error function vanishes, i.e., $e(S_f, S_{f_1}) = 0$. In other words, synchrosqueezing can theoretically separate *any* combination of two different harmonics if Δ can be picked sufficiently small.

However, in practice, Δ is fixed ahead of time; as shown in Sec. 3, the beating phenomenon then shows up when the frequencies of two composite harmonic functions are too close.

For fixed Δ, the case $\frac{1-\Delta}{1+\Delta} < \xi_0 < 1$ will arise for some ξ_0. Fix ξ_0 so that $\frac{1-\Delta}{1+\Delta} < \xi_0 < 1$. When $1-\Delta < a < 1+\Delta$, it follows that $\frac{(1-\Delta)^2}{1+\Delta} < a\xi_0 < (1+\Delta)$, which means $\omega_f(a,b)$ will contain frequency information from both f_1 and f_2. Indeed, after defining $\omega_f(a,b)$ over $Z(b) = \{(a,b) \in \mathbb{R}^2 : \hat{\psi}(a)e^{ib} + a\hat{\psi}(a\xi_0)e^{ib\xi_0} \neq 0\}$, we have

$$\begin{aligned} |\omega_f(a,b) - 1| &= \left| \frac{\hat{\psi}(a)e^{ib} + a\xi_0\hat{\psi}(a\xi_0)e^{ib\xi_0}}{\hat{\psi}(a)e^{ib} + a\hat{\psi}(a\xi_0)e^{ib\xi_0}} - 1 \right| \\ &= \left| \frac{a(\xi_0-1)\hat{\psi}(a\xi_0)e^{ib\xi_0}}{\hat{\psi}(a)e^{ib} + a\hat{\psi}(a\xi_0)e^{ib\xi_0}} \right| \\ &= \frac{a(1-\xi_0)\hat{\psi}(a\xi_0)}{\sqrt{a^2\hat{\psi}^2(a\xi_0) + \hat{\psi}^2(a) + 2a\hat{\psi}(a\xi_0)\hat{\psi}(a)\cos(b(1-\xi_0))}} \end{aligned}$$

Thus, synchrosqueezing gives

$$\begin{aligned}\Re S_f(1,b) &= \Re \int W_f(a,b) a^{-3/2} \delta(|\omega_f(a,b)-1|) da \\ &= \left[\int_{1-\Delta}^{1+\Delta} a^{-1}\hat{\psi}(a)\chi_{\left\{a<\frac{1-\Delta}{\xi_0}\right\}\cup\left\{a>\frac{1+\Delta}{\xi_0}\right\}} da\right] \cos 2\pi b \\ &\quad + \left[\int_{\frac{1-\Delta}{\xi_0}}^{\frac{1+\Delta}{\xi_0}} a^{-1}\hat{\psi}(a\xi_0)\chi_{\left\{a<\frac{1-\Delta}{\xi_0}\right\}\cup\left\{a>\frac{1+\Delta}{\xi_0}\right\}} da\right] a\cos 2\pi\xi_0 b \\ &= \left[\int a^{-1}\hat{\psi}(a)da\right]\cos 2\pi b - \left[\int_{\frac{1-\Delta}{\xi_0}}^{1+\Delta} a^{-1}\hat{\psi}(a)da\right]\cos 2\pi b.\end{aligned}$$

Similar argument shows that

$$|\omega_f(a,b)-\xi_0| = \frac{(1-\xi_0)\hat{\psi}(a)}{\sqrt{a^2\hat{\psi}^2(a\xi_0)+\hat{\psi}^2(a)+2a\hat{\psi}(a\xi_0)\hat{\psi}(a)\cos(b(1-\xi_0))}},$$

$$\Re S_f(\xi_0,b) = \left[\int a^{-1}\hat{\psi}(a)da\right] a\cos 2\pi\xi_0 b - \left[\int_{\frac{1-\Delta}{\xi_0}}^{1+\Delta} a^{-1}\hat{\psi}(a)da\right] a\cos 2\pi\xi_0 b.$$

This means that when $\frac{1-\Delta}{1+\Delta} < \xi_0 < 1$, $\Re S_f(1,b)$ is attenuated by the presence of f_2 and $\Re S_f(\xi_0,b)$ is attenuated by the presence of f_1. In particular, when $0 < 1-\xi_0 \ll 1$, $\Re S_f(1,b)$ and $\Re S_f(\xi_0,b)$ are both almost zero. This result explains why the error is small when $0 < \xi_0 - 1 \ll 1$ compared with other $\xi_1 \in (\frac{1-\Delta}{1+\Delta}, \xi_0)$ (see, e.g. Fig. 3). Moreover, the calculation shows that when two components have close frequencies, synchrosqueezing cannot separate them.

Note that since $\omega(a,b)$ is defined over $Z(b)$, we need not worry about whether the denominator of $|\omega_f(a,b)-1|$ is zero.

We can likewise consider $|\omega_f(a,b)-\xi|$ for $\xi \neq 1, \xi_0$:

$$\begin{aligned}|\omega_f(a,b)-\xi| &= \left|\frac{(1-\xi)\hat{\psi}(a)e^{ib} + a(\xi_0-\xi)\hat{\psi}(a\xi_0)e^{ib\xi_0}}{\hat{\psi}(a)e^{ib}+a\hat{\psi}(a\xi_0)e^{ib\xi_0}}\right| \\ &= \left|\frac{(1-\xi)\hat{\psi}(a)e^{ib(1-\xi_0)} + a(\xi_0-\xi)\hat{\psi}(a\xi_0)}{\hat{\psi}(a)e^{ib(1-\xi_0)}+a\hat{\psi}(a\xi_0)}\right|.\end{aligned}$$

We can have $|\omega_f(a,b)-\xi| = 0$ only if the numerator of the fraction is real, i.e., when $b = \frac{k\pi}{1-\xi_0}$, where $k \in \mathbb{Z}$. To observe what is going on, we take $\hat{\psi}(\xi) = e^{-|\xi-1|^2/\sigma}$, where σ is small, to facilitate the calculation. (To be a "true" wavelet, ψ should satisfy $\hat{\psi}(0) = 0$, which is not strictly the case here, but it is true for all practical purposes if σ is sufficiently small.) Thus, when $b = \frac{k\pi}{1-\xi_0}$, the numerator becomes $|(-1)^k(1-\xi)e^{-(a-1)^2/\sigma} + a(\xi_0-\xi)e^{-(a\xi_0-1)^2/\sigma}|$.

When k is even, we thus need to solve the following equation for a:

$$(1-\xi)e^{-(a-1)^2/\sigma} + a(\xi_0 - \xi)e^{-(a\xi_0-1)^2/\sigma} = 0. \tag{2}$$

This can have a solution only when $\xi_0 < \xi < 1$. (Recall that we consider only $\xi_0 < 1$; $\xi_0 > 1$ can be dealt with by similar argument.) We obtain $a = \frac{1-\xi_0 \pm \sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_e}}{1-\xi_0^2}$, where $C_e = \sigma \ln \frac{a(\xi-\xi_0)}{1-\xi}$; this has a real solution for a only when $(1-\xi_0)^2 + (\xi_0^2 - 1)C_e \geq 0$, that is, when $\xi \in (\xi_0, \big(1 + ae^{\frac{1-\xi_0}{\sigma(1+\xi_0)}}\big)^{-1}(1 + a\xi_0 e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}})] \equiv I_e$.

Denote $a_{e,1} = \frac{1-\xi_0-\sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_e}}{1-\xi_0^2}$ and $a_{e,2} = \frac{1-\xi_0+\sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_e}}{1-\xi_0^2}$ as the two solutions of a of Eq. (2) when $\xi \in I$. Note that since $\frac{1-\Delta}{1+\Delta} < \xi_0 < \xi < 1$ and $\Delta \ll 1$, we know $a_{e,1} < \frac{1}{1+\xi_0} < \frac{1}{1+\frac{1-\Delta}{1+\Delta}} = \frac{1+\Delta}{2} < 1-\Delta$. In conclusion, when $\xi \in I_e$, and $b = \frac{k\pi}{1-\xi_0}$, with k even, then $S_f(\xi, b)$ can be written down explicitly as

$$\begin{aligned} S_f(\xi,b) &= \int W_f(a,b)a^{-3/2}\delta(|\omega_f(a,b)-\xi|)da \\ &= \left[\int a^{-1}\hat{\psi}(a)\delta(|\omega_f(a,b)-\xi|)da\right]e^{i2\pi b} \\ &\quad + \left[\int a^{-1}\hat{\psi}(a\xi_0)\delta(|\omega_f(a,b)-\xi|)da\right]ae^{i2\pi\xi_0 b} \\ &= \frac{1}{a_{e,2}}\hat{\psi}(a_{e,2})e^{i\frac{k\pi}{1-\xi_0}} + \frac{a}{a_{e,2}}\hat{\psi}(a_{e,2}\xi_0)e^{i\frac{\xi_0 k\pi}{1-\xi_0}}. \end{aligned}$$

Similarly, when k is odd, we need to solve the equation

$$(\xi-1)e^{-(a-1)^2/\sigma} + a(\xi_0 - \xi)e^{-(a\xi_0-1)^2/\sigma} = 0. \tag{3}$$

In this case, the solution a exists when $\xi > 1$ or $\xi < \xi_0$. Similar argument gives us $a = \frac{1-\xi_0 \pm \sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_o}}{1-\xi_0^2}$, where $C_o = \sigma \ln \frac{a(\xi-\xi_0)}{\xi-1}$. The solution a is real when $a < e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}}$ and $\xi \in [(e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}} - a)^{-1}(e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}} - a\xi_0), \infty) \equiv I_{o,1}$, or when $a\xi_0 > e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}}$ and $\xi \in (0, (a - e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}})^{-1}(a\xi_0 - e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}})) \equiv I_{o,2}$.

Denote $a_{o,1} = \frac{1-\xi_0-\sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_o}}{1-\xi_0^2}$ and $a_{o,2} = \frac{1-\xi_0+\sqrt{(1-\xi_0)^2+(\xi_0^2-1)C_o}}{1-\xi_0^2}$ as the two solutions of a of Eq. (3) when $a < e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}}$ and $\xi \in I_{o,1}$, or when $a\xi_0 > e^{\frac{1-\xi_0}{\sigma(1+\xi_0)}}$ and $\xi \in I_{o,2}$. Note that $a_{o,1} < \frac{1}{1+\xi_0} < \frac{1}{1+\frac{1-\Delta}{1+\Delta}} = \frac{1+\Delta}{2} < 1-\Delta$. In conclusion, when $\xi \in I_{o,1}$ and $ae^{\frac{1-\xi_0}{\sigma(1+\xi_0)}} < 1$ (or when $\xi \in I_{o,2}$ and $ae^{\frac{1-\xi_0}{\sigma(1+\xi_0)}} > 1/\xi_0$), $b = \frac{k\pi}{1-\xi_0}$, k

odd, $S_f(\xi, b)$ can be written down explicitly as

$$S_f(\xi, b) = \frac{1}{a_{o,2}} \hat{\psi}(a_{o,2}) e^{i\frac{k\pi}{1-\xi_0}} + \frac{a}{a_{o,2}} \hat{\psi}(a_{o,2}\xi_0) e^{i\frac{\xi_0 k\pi}{1-\xi_0}}.$$

Putting together all the above explains what we see in Fig. 5: when $b = \frac{k\pi}{1-\xi_0}$, k even, $\text{supp}S_f(\xi, b) = \{\xi : S_f(\xi, b) \neq 0\} \subset I_e$, and when $b = \frac{k\pi}{1-\xi_0}$, k odd $\text{supp}S_f(\xi, b) = \{\xi : S_f(\xi, b) \neq 0\} \subset I_{o,1}$ or $I_{o,2}$ depending on a. Since $I_{o,1} \cap I_e = \emptyset$ and $I_{o,2} \cap I_e = \emptyset$, we conclude that when $\xi_0 > \frac{1-\Delta}{1+\Delta}$, $S_f(\xi, b)$ exhibits an oscillating pattern, which prevents us from telling either the signal is composed of one or two components. Note that the spreading of the support $\text{supp}S_f(\xi, b)$ depends on not only ξ_0, but also on a and σ.

5. Conclusion

This study has considered how synchrosqueezing behaves in signals consisting of two harmonic functions with different frequencies and amplitudes, i.e., $f(t) = f_1(t) + f_2(t) = \cos(2\pi t) + a\cos(2\pi\xi t)$. Since synchrosqueezing is a nonlinear and adaptive method that captures some of the features of EMD, it is interesting to assess in what aspects they differ and how much they share. Guided by the signal, synchrosqueezing gives an analysis that is as intuitive as that given by EMD. However, the difference of synchrosqueezing and EMD can be noticed in the result of applying them to $f(t) = f_1(t) + f_2(t)$. In particular, the reaction of synchrosqueezing is different from that of EMD, especially when the amplitude a is small, i.e., the transition from zones 1 to 2 in Fig. 4; when the amplitude a is larger, there is no sharp transition as in EMD as can be seen in the transition area from zone 1 to zone 3 in Fig. 4. Moreover, as predicted in Sec. 4, the smaller the support of $\hat{\psi}$, the more accurate the separation of the two components. Thus, we can answer the questions we asked in the beginning: when $\xi_0 \leq \frac{1-\Delta}{1+\Delta}$, where $\text{supp}\hat{\psi} \subset [1-\Delta, 1+\Delta]$, synchrosqueezing can retrieve the two individual tones, otherwise synchrosqueezing gets confused.

Compared with EMD, which highly depends on local extremes, synchrosqueezing depends on the continuous wavelet transform and reassignment. This partially explains why the result of synchrosqueezing and EMD is different when applied to $f(t) = f_1(t) + f_2(t)$. One resource of the beating of synchrosqueezing is its ability to extract nonharmonic component. This can be seen clearly through the calculus of variation interpretation provided in [Daubechies *et al.* (2009)]. The calculus of variation interpretation says that if the given function f is in the proper function class $\mathcal{A}_{\epsilon,d}$, which is composed of finite well separated intrinsic mode type functions, then its synchrosqueezed wavelet transform $S_f(\xi, b)$ approximates the minimizer of the following functional

$$\int \left| \mathfrak{Re}\left[\int F(\xi, b) d\xi \right] - f(b) \right|^2 \mathrm{d}b + \mu \iint |\partial_b F(\xi, b) - i\xi F(\xi, b)|^2 \,\mathrm{d}b\mathrm{d}\xi. \qquad (4)$$

The motivations of this functional come from reconstructing the given function from its decomposition and the following relationship

$$\partial_t e^{i\xi t} = i\xi e^{i\xi t}, \tag{5}$$

A close look into Eq. (4) tells us that the second term helps to relax the frequency notion in Fourier transform. It is this relaxation that help synchrosqueezing to extract intrinsic mode type "building block" functions from a given function in the proper function class $\mathcal{A}_{\epsilon,d}$. Further, it is this relaxation that generates the beating phenomenon.

The model analyzed here is of course oversimplified, so as to make it feasible to compute $|\omega(a,b)-\xi| = 0$; nevertheless, this simplified model helps us to understand more details of the synchrosqueezing algorithm.

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