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MULTI-FUZZY EXTENSIONS OF FUNCTIONS

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In this paper, we study various properties of multi-fuzzy extensions of crisp functions using order homomorphisms, complete lattice homomorphisms, L-fuzzy lattices, and strong L-fuzzy lattices as bridge functions.

Keywords: Multi-fuzzy set; bridge function; multi-fuzzy extension.

1. Introduction

We proposed the theory of multi-fuzzy sets [Sabu and Ramakrishnan (2010a), (2011a)] as an extension of theories of fuzzy sets, L-fuzzy sets [Goguen (1967)] and intuitionistic fuzzy sets [Atanassov (1986)]. Theory of multi-fuzzy sets deals with multi-level fuzziness and multi-dimensional fuzziness. Our previous papers discussed the basic notions of multi-fuzzy sets, multi-fuzzy topology, and multi-fuzzy subgroups [Sabu and Ramakrishnan (2010a), (2010c), (2011b) (2011a)]. Bridge functions and multi-fuzzy extensions of crisp functions have great importance in the study of multi-fuzzy sets. In this paper, multi-fuzzy extensions of crisp functions based on the bridge functions such as order homomorphisms, complete lattice homomorphisms, L-fuzzy lattices, and strong L-fuzzy lattices are studied.

2. Preliminaries

We use the following notations. X and Y stand for universal sets, I, J, and K stand for indexing sets, L and M stand for partially ordered sets, L^X stands for the set of all functions from X to L, $\{L_j : j \in J\}$, and $\{M_i : i \in I\}$ stand for families of complete lattices with order reversing involutions, unless it is stated otherwise. Partial order $\geq$ is the opposite order relation of the partial

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order $\leq$. The products $\prod M_i, \prod L_j, \prod M_i^X$, and $\prod L_j^Y$ stand for the products $\prod_{i \in I} M_i$, $\prod_{j \in J} L_j$, $\prod_{i \in I} M_i^X$, and $\prod_{j \in J} L_j^Y$, respectively.

Definition 1. [Goguen (1967)] Let X be a nonempty ordinary set and L be a complete lattice. An L-fuzzy set on X is a mapping $A : X \to L$, that is, the family of all the L-fuzzy sets on X is just L^X consisting of all the mappings from X to L.

Definition 2. [Wang (1984)] Let $' : M \to M$ and $' : L \to L$ be order reversing involutions. A mapping $h : M \to L$ is called an order homomorphism, if it satisfies the conditions $h(0) = 0$, $h(\vee a_i) = \vee h(a_i)$, and $h^{-1}(b') = (h^{-1}(b))'$.

$h^{-1} : L \to M$ is defined by $\forall\, b \in L$, $h^{-1}(b) = \vee\{a \in M : h(a) \leq b\}$. Wang [1984] proved the following properties of order homomorphism. For every $a \in M$ and $p \in L$; $a \leq h^{-1}(h(a))$, $h(h^{-1}(p)) \leq p$, $h^{-1}(1_L) = 1_M$, $h^{-1}(0_L) = 0_M$, and $a \leq h^{-1}(p)$ if and only if $h(a) \leq p$ if and only if $h^{-1}(p') \leq a'$. Both h and h^{-1} are order preserving and arbitrary join preserving maps. Moreover, $h^{-1}(\wedge a_i) = \wedge h^{-1}(a_i)$.

Definition 3. (see [Ying-Ming and Mao-Kang (1997)]) If $\{L_j : j \in J\}$ is a family of lattices, then the product $\prod L_j$ is a lattice if for arbitrary $x, y \in \prod L_j$, the join $x \vee y$ and the meet $x \wedge y$ of x, y are defined as:

$$(x \vee y)_j = x_j \vee y_j \quad \text{and} \quad (x \wedge y)_j = x_j \wedge y_j,\ \forall\ x_j, y_j \in L_j,\ \forall\, j \in J;$$

or, equivalently, $x \leq y$ is defined by $x_j \leq_j y_j$, $\forall j \in J$, where $\leq$ and $\leq_j$ are the order relations in $\prod L_j$ and L_j, respectively.

2.1. *L-fuzzy lattices and strong L-fuzzy lattices*

Definition 4. [Tepavčević and Trajkovski (2001)] Let $(M, \wedge_M, \vee_M)$ be a lattice and L be a complete lattice with the least element 0_L and the greatest element 1_L. The mapping $A : M \to L$ is called a lattice-valued fuzzy lattice (L-fuzzy lattice) if all the p-level sets ($p \in L$) of A are sublattices of M.

Lemma 1. [Tepavčević and Trajkovski (2001)] *Let $(M, \wedge_M, \vee_M)$ be a lattice and $(L, \wedge_L, \vee_L)$ a complete lattice with 0_L and 1_L.*

(1) *Let $p, q \in L$ and $A : M \to L$ be an L-fuzzy lattice. If $p \leq q$, then the q-level set $A_q = \{x \in M : q \leq A(x)\}$ is a sublattice of the p-level set $A_p = \{x \in M : p \leq A(x)\}$.*
(2) *A mapping $A : M \to L$ is an L-fuzzy lattice if and only if $A(x) \wedge_L A(y) \leq A(x \wedge_M y)$ and $A(x) \wedge_L A(y) \leq A(x \vee_M y)$, for all $x, y \in M$.*

Definition 5. [Sabu and Ramakrishnan (2010b)] Let $(M, \wedge, \vee)$ be a lattice and $(L, \wedge, \vee)$ be a lattice with the least element 0_L and the greatest element 1_L. The mapping $A : M \to L$ is called a strong L-fuzzy lattice if $A_a^b = \{x \in M : a \leq A(x) \leq b\}$ is a sublattice of M, for all $a, b \in L$.

Lemma 2. [Sabu and Ramakrishnan (2010b)] *Let $A : M \to L$ be a strong* L*-fuzzy lattice, and let $p, q, r, s \in L$. If $p \leq q \leq r \leq s$, then A_q^r is a sublattice of A_p^s.*

Theorem 1. [Sabu and Ramakrishnan (2010b)] *Let $(M, \wedge, \vee)$ be a lattice and $(L, \wedge, \vee)$ a complete lattice with 0_L and 1_L. The mapping $A : M \to L$ is a strong* L*-fuzzy lattice if and only if A satisfies the following conditions, for all $x, y \in M$:*

(1) $A(x) \wedge A(y) \leq A(x \wedge y) \leq A(x) \vee A(y)$; *and*
(2) $A(x) \wedge A(y) \leq A(x \vee y) \leq A(x) \vee A(y)$.

Theorem 2. [Sabu and Ramakrishnan (2010b)] *Let M be a lattice, L be a complete lattice and $A : M \to L$ be a mapping. A is a lattice homomorphism if and only if A is an order preserving strong* L*-fuzzy lattice.*

2.2. *Multi-fuzzy sets*

Definition 6. [Sabu and Ramakrishnan (2010a), (2011a)] Let X be a nonempty set, J be an indexing set, and $\{L_j : j \in J\}$ a family of partially ordered sets. A **multi-fuzzy set** A in X is a set:

$$A = \{\langle x, (\mu_j(x))_{j \in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}. \tag{1}$$

The function $\mu_A = (\mu_j)_{j \in J}$ is called the multi-membership function of the multi-fuzzy set A. Complement of A is $A' = \{\langle x, (\mu_j'(x))_{j\in J}\rangle : x \in X\}$, where μ_j' is the order reversing involution of μ_j.

Definition 7. [Sabu and Ramakrishnan (2010a)] Let $\{L_j : j \in J\}$ be a family of partially ordered sets,

$$A = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X, \mu_j \in L_j^X, j \in J\} \quad \text{and}$$
$$B = \{\langle x, (\nu_j(x))_{j\in J}\rangle : x \in X, \nu_j \in L_j^X, j \in J\}$$

be multi-fuzzy sets in a nonempty set X with the product order, then $A \sqsubseteq B$ if and only if $\mu_j(x) \leq \nu_j(x)$, $\forall\, x \in X$ and $\forall\, j \in J$.

The equality, union, and intersection of A and B are defined as:

(1) $A = B$ if and only if $\mu_j(x) = \nu_j(x)$, $\forall\, x \in X$ and $\forall\, j \in J$;
(2) $A \sqcup B = \{\langle x, (\mu_j(x) \vee \nu_j(x))_{j\in J}\rangle : x \in X\}$; and
(3) $A \sqcap B = \{\langle x, (\mu_j(x) \wedge \nu_j(x))_{j\in J}\rangle : x \in X\}$.

3. Bridge Functions and Multi-Fuzzy Extensions

Mappings from $\prod M_i^X$ into $\prod L_j^Y$ have fundamental role in the study of multi-fuzzy sets. In this section, we introduce the notion of multi-fuzzy extension of crisp functions, which is useful to connect multi-fuzzy sets with different value domains and different dimensions.

Definition 8. [Sabu and Ramakrishnan (2011a)] Let $f: X \to Y$ and $h: \prod M_i \to \prod L_j$ be functions. The multi-fuzzy extension $F: \prod M_i^X \to \prod L_j^Y$ of f with respect to h is defined by

$$F(A)(y) = \bigvee_{y=f(x)} h(A(x)), \quad A \in \prod M_i^X, \quad y \in Y \tag{2}$$

and its inverse $F^{-1}: \prod L_j^Y \to \prod M_i^X$ is defined by

$$F^{-1}(B)(x) = h^{-1}(B(f(x))), \quad B \in \prod L_j^Y, \quad x \in X; \tag{3}$$

where h^{-1} is the upper adjoint of h in Wang's [1984] sense (see Definition 2). The lattice valued function $h: \prod M_i \to \prod L_j$ is called the **bridge function** of the multi-fuzzy extension of f.

Order homomorphisms, lattice homomorphisms, arbitrary join preserving maps, complement preserving maps, etc. are useful bridge functions for multi-fuzzy extensions.

3.1. *Extensions based on order homomorphisms*

Lemma 3. [Sabu and Ramakrishnan (2011a)] *If an order homomorphism $h: \prod M_i \to \prod L_j$ is the bridge function for the multi-fuzzy extension of a crisp function $f: X \to Y$, then for any $k \in K$, $A_k \in \prod M_i^X$, $B_k \in \prod L_j^Y$:*

(1) $f(0_X) = 0_Y$;
(2) $f(\sqcup A_k) = \sqcup f(A_k)$; *and*
(3) $(f^{-1}(B))' = f^{-1}(B')$,

that is, the extension map f is an order homomorphism.

Theorem 3. [Sabu and Ramakrishnan (2011a)] *If an order homomorphism $h: \prod M_i \to \prod L_j$ is the bridge function for the multi-fuzzy extension of a crisp function $f: X \to Y$, then for any $k \in K$, $A_k \in \prod M_i^X$, $B_k \in \prod L_j^Y$:*

(1) $A_1 \sqsubseteq A_2$ *implies* $f(A_1) \sqsubseteq f(A_2)$;
(2) $f(\sqcup A_k) = \sqcup f(A_k)$;
(3) $f(\sqcap A_k) \sqsubseteq \sqcap f(A_k)$;
(4) $f(A_{[\alpha]}) \subseteq f(A)_{[h(\alpha)]}$;
(5) $f^{-1}(1_Y) = 1_X$ *and* $f^{-1}(0_Y) = 0_X$;
(6) $B_1 \sqsubseteq B_2$ *implies* $f^{-1}(B_1) \sqsubseteq f^{-1}(B_2)$;
(7) $f^{-1}(\sqcup B_k) = \sqcup f^{-1}(B_k)$;
(8) $f^{-1}(\sqcap B_k) = \sqcap f^{-1}(B_k)$;
(9) $A \sqsubseteq f^{-1}(f(A))$; *and*
(10) $f(f^{-1}(B)) \sqsubseteq B$.

3.2. *Extensions based on complete lattice homomorphisms*

Theorem 4. *If a complete lattice homomorphism $h\colon \prod M_i \to \prod L_j$ is the bridge function for the multi-fuzzy extension of a crisp function $f\colon X \to Y$, then for any $k \in K$, $A_k \in \prod M_i^X$, $B_k \in \prod L_j^Y$:*

(1) $f(0_X) = 0_Y$ *and* $f(1_X) = 1_Y$;
(2) $A_1 \sqsubseteq A_2$ *implies* $f(A_1) \sqsubseteq f(A_2)$;
(3) $f(\sqcup A_k) = \sqcup f(A_k)$;
(4) $f(\sqcap A_k) \sqsubseteq \sqcap f(A_k)$;
(5) $f^{-1}(1_Y) = 1_X$;
(6) $B_1 \sqsubseteq B_2$ *implies* $f^{-1}(B_1) \sqsubseteq f^{-1}(B_2)$;
(7) $\sqcup f^{-1}(B_k) \sqsubseteq f^{-1}(\sqcup B_k)$; *and*
(8) $f^{-1}(\sqcap B_k) \sqsubseteq \sqcap f^{-1}(B_k)$.

Proof. Similar to that of Theorem 3. □

3.3. *L-fuzzy lattice and strong L-fuzzy lattice extensions*

Theorem 5. *Let $\{L_j : j \in J\}$ and $\{M_i : i \in I\}$ be families of completely distributive lattices and the strong lattice valued fuzzy lattice $h\colon \prod M_i \to \prod L_j$ be the bridge function for the multi-fuzzy extension of $f\colon X \to Y$. Then, the multi-fuzzy extension $f\colon \prod M_i^X \to \prod L_j^Y$ of f is a strong lattice valued fuzzy lattice with respect to the inclusion as the order relation.*

Proof. For every $y \in Y$,

$$\begin{aligned}
f(A_1 \sqcup A_2)(y) &= \bigvee\{h((A_1 \sqcup A_2)(x)) : x \in X,\ y = f(x)\}\\
&= \bigvee\{h(A_1(x) \vee A_2(x)) : x \in X,\ y = f(x)\}\\
&\geq \bigvee\{h(A_1(x)) \wedge h(A_2(x)) : x \in X,\ y = f(x)\}\\
&= \left(\bigvee\{h(A_1(x)) : x \in X, y = f(x)\}\right)\\
&\quad \wedge \left(\bigvee\{(h(A_2(x)) : x \in X, y = f(x)\}\right)\\
&= f(A_1)(y) \wedge f(A_2)(y)\\
&= (f(A_1) \sqcap f(A_2))(y).
\end{aligned}$$

That is,

$$f(A_1) \sqcap f(A_2) \sqsubseteq f(A_1 \sqcup A_2). \tag{4}$$

$$\begin{aligned}
f(A_1 \sqcap A_2)(y) &= \bigvee\{h((A_1 \sqcap A_2)(x)) : x \in X,\ y = f(x)\}\\
&= \bigvee\{h(A_1(x) \wedge A_2(x)) : x \in X, y = f(x)\}
\end{aligned}$$

$$\begin{aligned}
&\geq \bigvee\{h(A_1(x)) \wedge h(A_2(x)) : x \in X,\ y = f(x)\}\\
&= \Big(\bigvee\{h(A_1(x)) : x \in X,\, y = f(x)\}\Big)\\
&\quad \wedge \Big(\bigvee\{(h(A_2(x)) : x \in X,\, y = f(x)\}\Big)\\
&= f(A_1)(y) \wedge f(A_2)(y)\\
&= (f(A_1) \sqcap f(A_2))(y).
\end{aligned}$$

That is,

$$f(A_1) \sqcap f(A_2) \sqsubseteq f(A_1 \sqcap A_2). \tag{5}$$

Equations (4) and (5) together imply, the multi-fuzzy extension of f is a lattice valued fuzzy lattice with respect to the inclusion as the order relation. For every $y \in Y$,

$$\begin{aligned}
f(A_1 \sqcup A_2)(y) &= \bigvee\{h((A_1 \sqcup A_2)(x)) : x \in X,\, y = f(x)\}\\
&= \bigvee\{h(A_1(x) \vee A_2(x)) : x \in X,\, y = f(x)\}\\
&\leq \bigvee\{h(A_1(x)) \vee h(A_2(x)) : x \in X,\, y = f(x)\}\\
&= \Big(\bigvee\{h(A_1(x)) : x \in X,\, y = f(x)\}\Big)\\
&\quad \vee \Big(\bigvee\{(h(A_2(x)) : x \in X,\, y = f(x)\}\Big)\\
&= f(A_1)(y) \vee f(A_2)(y)\\
&= (f(A_1) \sqcup f(A_2))(y).
\end{aligned}$$

That is,

$$f(A_1 \sqcup A_2) \sqsubseteq f(A_1) \sqcup f(A_2). \tag{6}$$

$$\begin{aligned}
f(A_1 \sqcap A_2)(y) &= \bigvee\{h((A_1 \sqcap A_2)(x)) : x \in X,\, y = f(x)\}\\
&= \bigvee\{h(A_1(x) \wedge A_2(x)) : x \in X,\, y = f(x)\}\\
&\leq \bigvee\{h(A_1(x)) \vee h(A_2(x)) : x \in X,\, y = f(x)\}\\
&= \Big(\bigvee\{h(A_1(x)) : x \in X,\, y = f(x)\}\Big)\\
&\quad \vee \Big(\bigvee\{(h(A_2(x)) : x \in X,\, y = f(x)\}\Big)\\
&= f(A_1)(y) \vee f(A_2)(y)\\
&= (f(A_1) \sqcup f(A_2))(y).
\end{aligned}$$

That is,

$$f(A_1 \sqcap A_2) \sqsubseteq f(A_1) \sqcup f(A_2). \tag{7}$$

Hence, Eqs. (4) and (6) together imply,

$$f(A_1) \sqcap f(A_2) \sqsubseteq f(A_1 \sqcup A_2) \sqsubseteq f(A_1) \sqcup f(A_2), \tag{8}$$

also Eqs. (5) and (7) together imply,

$$f(A_1) \sqcap f(A_2) \sqsubseteq f(A_1 \sqcap A_2) \sqsubseteq f(A_1) \sqcup f(A_2). \tag{9}$$

Equations (8) and (9) together imply, the multi-fuzzy extension of f is a strong lattice valued fuzzy lattice. □

Corollary 1. *Let $\{L_j : j \in J\}$ and $\{M_i : i \in I\}$ be families of completely distributive lattices, lattice-valued fuzzy lattice $h : \prod M_i \to \prod L_j$ be the bridge function for the multi-fuzzy extension of $f : X \to Y$. The multi-fuzzy extension $f : \prod M_i^X \to \prod L_j^Y$ of f is a lattice-valued fuzzy lattice with respect to the inclusion as the order relation.*

Lemma 4. *Let $f : \prod M_i^X \to \prod L_j^Y$ be the multi-fuzzy extension of a crisp function $f : X \to Y$ with respect to the lattice-valued fuzzy lattice $h : \prod M_i \to \prod L_j$ as the bridge function. Then*:

(1) *$f(0_X)$ need not be equal to 0_Y.*
(2) *$f^{-1}(0_Y)$ need not be equal to 0_X.*
(3) *$A \sqsubseteq B$ need not implies $f(A) \sqsubseteq f(B)$.*

Proof. We prove these results by a counter example. Let $L = \{0_L, 1_L\}$, $M = \{0_M, a, b, 1_M\}$ be the diamond lattice with the least element 0_M and the greatest element 1_M. Suppose the L-fuzzy lattice $h : M \to L$ defined by, for $m \in M$:

m	0_M	a	b	1_M
$h(m)$	1_L	0_L	0_L	1_L

is the bridge function for the multi-fuzzy extension $f : M^X \to L^Y$ of the crisp function $f : X \to Y$.

(1) For every $y \in Y$,

$$f(0_X)(y) = \vee\{h(0_X(x)) : x \in X,\ y = f(x)\} = h(0_M) = 1_L.$$

Hence, $1_Y = f(0_X) \neq 0_Y$.

(2) For every $x \in X$,

$$f^{-1}(0_Y)(x) = h^{-1}(0_Y(f(x))) = h^{-1}(0_L) = a \vee b = 1_M.$$

Therefore, $f^{-1}(0_Y) = 1_X \neq 0_X$.

(3) Let $A(x) = 0_M, B(x) = a, \forall x \in X$. $A \sqsubseteq B$; Since,

$$0_M = A(x) \leq B(x) = a, \forall x \in X.$$

However, $h(A(x)) = h(0_M) = 1_L$ and $h(B(x)) = h(a) = 0_L$. Hence, $h(A(x)) \nleq h(B(x))$ and so $f(A) \not\sqsubseteq f(B)$. □

4. Image Processing Using Multi-Fuzzy Sets

A digital image is a representation of a two-dimensional (2D) image of finite set of digital values, called pixels (or picture elements). Pixel is the smallest addressable area of a display. Any image can be approximated by an $m \times n$ matrix of pixels. The resolution of an image is described as the number of pixels horizontally times the number of pixels vertically. We will refer to a pixel by the number of its row and the number of its column. By this convention, the x-axis is vertical and the y-axis is horizontal. Pixel values typically represent gray levels, colors, heights, etc. Digital image processing focuses on two major tasks: improvement of pictorial information for human interpretation and processing of image data for storage, transmission, and representation for autonomous machine perception (see [Plataniotis *et al.* (1995); (1998)].

The primary colors red, green, and blue are combined to reproduce other colors. Color image are usually described in the additive color space RGB. In the RGB color space, a color is represented by a triplet (R, G, B), where R, G, and B denote the intensities of the red component, the green component, and the blue component, respectively. Note that, the intensities of the pixels are integers in the interval $[0, p]$ (usually $p = 255$). Digital form of a color image, is a mapping from a 2D grid of uniformly spaced discrete points $\{(i, j) : i \in \mathbb{N}_m, j \in \mathbb{N}_n\}$ into $[0, p]^3$, where $\mathbb{N}_m$ is the set of all positive integers less than or equal to m. That is, $(R(i, j), G(i, j), B(i, j))$ is the $p + 1$-scale color value of the pixel (i, j).

Theory of multi-fuzzy sets has practical relevance in image processing. In multi-fuzzy image processing method, the pixel value $(R(i, j), G(i, j), B(i, j))$ of the given image is converted into its multi-fuzzy counterpart, $(r(i, j), g(i, j), b(i, j))$, for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. See one of the ordered triple of useful conversion formulae:

$$r(i, j) = \left(1 + \frac{\bar{r} - R(i, j)}{\rho}\right)^{-u} \tag{10}$$

$$g(i, j) = \left(1 + \frac{\bar{g} - G(i, j)}{\gamma}\right)^{-v} \tag{11}$$

$$b(i, j) = \left(1 + \frac{\bar{b} - B(i, j)}{\beta}\right)^{-w}, \tag{12}$$

where $\bar{r}, \bar{g}, \bar{b} \in [0, p]$ are the reference constants defining the degrees of brightness and $\rho, \gamma, \beta, u, v,$ and w are the appropriate positive real numbers for the conversion (see the fuzzy image-processing problem in the book by [Klir and Yuan (1995)].)

Color of a pixel cannot be represented by a membership function of an ordinary fuzzy set, but it is possible to characterize by a 3D membership function (μ_r, μ_g, μ_b); where μ_r, μ_g and μ_b are the membership functions from $\{1, \ldots, m\} \times \{1, \ldots, n\}$ into $[0, 1]$. Therefore, a color image can be approximated by a collection of pixels with a multi-membership function (μ_r, μ_g, μ_b). The multi-fuzzy set $\{\langle x, \mu_1(x), \mu_2(x), \mu_3(x)\rangle : x = (i, j) \in X,\ \mu_1(x) = r(i, j),\ \mu_2(x) = g(i, j),\ \mu_3(x) = b(i, j),\ i \in \mathbb{N}_m, j \in \mathbb{N}_n\}$ represents the color image.

4.1. *How to reconstruct a color image from three linearly independent gray images of a picture?*

Using 3D multi-fuzzy membership functions, we can characterize the color of a pixel in an image. Suppose an image is approximated by an $m \times n$ matrix of pixels with multi-fuzzy membership function (μ_r, μ_g, μ_b). In this problem, we choose $p = 255$ and the membership values $\mu_r(i,j), \mu_g(i,j), \mu_b(i,j)$ as the normalized red value, green value, and blue value of the pixel (i,j). That is, $\mu_r(i,j) = R(i,j)/255$, $\mu_g(i,j) = G(i,j)/255$, and $\mu_b(i,j) = B(i,j)/255$. Let X be the set consisting of the mn pixels. Therefore, the color image can be approximated by the collection of pixels with the multi-membership function (μ_r, μ_g, μ_b) and it can be represented as a multi-fuzzy set

$$A = \{\langle x, \mu_r(x), \mu_g(x), \mu_b(x)\rangle : x = (i,j),\ (i,j) \in X\}. \tag{13}$$

Construct three different gray images of the color image A as follows. Let x be an arbitrary pixel of the image, for $k, l = 1, 2, 3$; $a_{(k,l)} \geq 0$ and $a_{(k,1)} + a_{(k,2)} + a_{(k,3)} = 1$; using the bridge functions $h_k : [0,1]^3 \to [0,1]$ define the gray values $G_1(x), G_2(x)$ and $G_3(x)$ as

$$G_k(x) = a_{(k,1)}\mu_r(x) + a_{(k,2)}\mu_g(x) + a_{(k,3)}\mu_b(x), \quad \text{for } k = 1, 2, 3; \tag{14}$$

where $h_k(a,b,c) = a_{(k,1)}a + a_{(k,2)}b + a_{(k,3)}c$. The fuzzy sets $G_k = \{\langle x, G_k(x)\rangle : x \in X\}$; (for $k = 1, 2, 3$) are the gray images (with different gray tones) of the color image A. If the coefficient matrix of $G_k(x)$, for $k = 1, 2, 3$ (the coefficient matrix of the system of linear Eq. (14)) is invertible; then, by using matrix inversion, we can reconstruct the original color image from the three gray images. Let $PM(x) = H(x)$ be the matrix representation of the pixel $x \in X$ of the three gray images, where

$$P = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \end{bmatrix},$$

the transpose of $M(x) = (\mu_r(x), \mu_g(x), \mu_b(x)) = \mu_A(x)$, and the transpose of $H(x) = (G_1(x), G_2(x), G_3(x))$.

If P is invertible, then we have $M(x) = P^{-1}H(x)$. Hence, we can reconstruct the color membership value $\mu_A(x) = (\mu_r(x), \mu_g(x), \mu_b(x))$ of the pixel x from three linearly independent gray values. See the following example. Consider the color image 'Bird' with the size 540×415 pixels and put

$$G_1(x) = 0.299\mu_r(x) + 0.587\mu_g(x) + 0.114\mu_b(x),$$

$$G_2(x) = 0.114\mu_r(x) + 0.299\mu_g(x) + 0.587\mu_b(x),$$

$$G_3(x) = 0.587\mu_r(x) + 0.114\mu_g(x) + 0.299\mu_b(x).$$

The fuzzy sets G_1, G_2, and G_3 represent the gray images: Gray image-1 (Fig. 1), Gray image-2 (Fig. 2) and Gray image-3 (Fig. 3) respectively. Figures 4 and 5 are

Fig. 1. Gray image-1 of bird.

Fig. 2. Gray image-2 of bird.

Fig. 3. Gray image-3 of bird.

Fig. 4. Original image of bird.

Fig. 5. Reconstructed image of bird.

the original and reconstructed images respectively. The matrices of the pixel values of these images are very large; therefore, we do not present the matrices.

Photo-credit (of the original image 'Bird') goes to Nobert Soloria Bermosa, The 20 most brilliantly colored birds in the world — Painted Bunting (Passerina Ciris), http://scienceray.com/biology/zoology. Each of the gray images, the original image and the reconstructed image has 540×415 pixels. File sizes of the original image, the reconstructed image, Gray image-1, Gray image-2, and Gray image-3 are 371 KB, 42 KB, 35 KB, 35 KB, and 35 KB, respectively. Gray images and the reconstructed image are prepared by using MATLAB 7.10. Average time taken for the formation of reconstructed image from the gray images is 0.0781 Sec on a system with Core 2 Duo 2.8 Ghz CPU and 2 GB RAM

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