

ON INTRINSIC MODE FUNCTION

GANG WANG*, XIAN-YAO CHEN† and FANG-LI QIAO‡

*The First Institute of Oceanography, State Oceanic Administration,
Key Laboratory of Marine Science and Numerical
Modeling (MASNUM), State Oceanic Administration,
Qingdao, 266061, P. R. China*

*wangg@fio.org.cn

†chenxy@fio.org.cn

‡qiaofl@fio.org.cn

ZHAOHUA WU

*Department of Meteorology,
Florida State University,
Tallahassee, FL 32306 USA
Zwu@fsu.edu*

NORDEN E. HUANG

*Research Center for Adaptive Data Analysis,
National Central University,
Chungli, Taiwan 34001
Norden@ncu.edu.tw*

Empirical Mode Decomposition (EMD) has been widely used to analyze non-stationary and nonlinear signal by decomposing data into a series of intrinsic mode functions (IMFs) and a trend function through sifting processes. For lack of a firm mathematical foundation, the implementation of EMD is still empirical and *ad hoc*. In this paper, we prove mathematically that EMD, as practiced now, only gives an approximation to the true envelope. As a result, there is a potential conflict between the strict definition of IMF and its empirical implementation through natural cubic spline. It is found that the amplitude of IMF is closely connected with the interpolation function defining the upper and lower envelopes: adopting the cubic spline function, the upper (lower) envelope of the resulting IMF is proved to be a unitary cubic spline line as long as the extrema are sparsely distributed compared with the sampling data. Furthermore, when natural spline boundary condition is adopted, the unitary cubic spline line degenerates into a straight line. Unless the amplitude of the IMF is a strictly monotonic function, the slope of the straight line will be zero. It explains why the amplitude of IMF tends to be a constant with the number of sifting increasing *ad infinitum*. Therefore, to get physically meaningful IMFs the sifting times for each IMF should be kept low as in the practice of EMD. Strictly speaking, the resolution of these difficulties should be either to change the EMD implementation method and eschew the spline, or to define the stoppage criterion more objectively and leniently. Short of the full resolution of the conflict, we should realize that the EMD as implemented now yields an approximation with respect to cubic

spline basis. We further concluded that a fixed low number of iterations would be the best option at this time, for it delivers the best approximation.

Keywords: Intrinsic mode function; cubic spline function; upper and lower envelopes.

1. Introduction

Intrinsic Mode Function (IMF) was introduced by Huang *et al.* [1998] as the result of the Empirical Mode Decomposition (EMD). It is a necessary intermediate step toward computing instantaneous frequency through the Hilbert Transform or any other methods [Huang *et al.* 2009]. Therefore, it is a key part of the Hilbert Spectral Analysis in the NASA designated Hilbert-Huang Transform (HHT). Since its introduction, HHT has found a wide range of applications: voice [Khaldi *et al.* (2010)], image [Nunes and Deléchelle (2009); Linderhed (2009); Chen *et al.* (2008); Sinclair and Pegram (2005) and Wu *et al.* (2009)], medicine [Chen (2009); Nunes *et al.* (2005); Liang *et al.* (2005)], climate or atmosphere [Ruzmaikin and Feynman (2009); Huang *et al.* (2009); Molla *et al.* (2007); Iyengar and Kanth (2006), Wu *et al.* (2008), Qian *et al.* (2009)], gravitational wave [Camp *et al.* (2009)], ocean [Huang and Wu (2008); Huang *et al.* (1999)], and geography [Wilson *et al.* (2007); Battista *et al.* (2007); Zhang *et al.* (2003)]. Many of the applications are actually based on IMFs. As a result, in the subsequent development of HHT, most of the efforts had been concentrated in the improvements of EMD, such as the intermittence test [Huang *et al.*, 1999; 2003], Ensemble EMD [EEMD, Wu and Huang, 2009], Complementary EEMD [CEEMD, Yeh *et al.* (2010)]. The EMD method has recently also been extended to multi-dimensional data by Wu *et al.* [2009] and others [Shi *et al.* (2009), Fauchereau *et al.* (2008), Liu *et al.* (2007)].

With these advances, the implementation of EMD seemed to have satisfied the requirements of most practical applications. Left untreated, however, is the rigorous mathematic foundation, which is absolutely necessary for making the empirical approach standing on a more solid foundation. Unfortunately, the progress is painfully slow. Among the urgently needed definitive work are on the definition of IMF and the stoppage criterion for EMD. Indeed, the recent study by Wu and Huang [2010] has shown empirically that the sifting process used in EMD could be equivalent to a bank of dyadic filter only when the number of sifting iterations is fixed to ten. The ratio of mean frequency between neighboring components would decrease below 2 as the numbers of iterations increase. In the limit when EMD is carried toward the limit with infinite many iteration of sifting, the ratio for mean frequency between neighboring components would approach unit. Then, EMD would produce IMFs of constant amplitude. Even though the limiting case actually conforms to the definition of IMFs better, such results would produce frequency modulated (FM) functions for IMFs, which might even approach Fourier expansion and lost all their intrinsic physical significance. Based on this empirical study, they suggested a fixed number of sifting for EMD implementation.

The observations by Wu and Huang [2010] suggest that there might be a conflict between the definition of IMF and the method proposed to implement it. To clarify these potential conflicts and to support the empirical approach of Wu and Huang [2010], we have embarked on the present study. In this paper, we will first revisit the EMD procedure, as originally introduced by [Huang *et al.* (1998)], and the definition of IMF. Then, a theorem would be proven to highlight the limitation and the consequence of the spline approach. By this theorem, we hope we could arouse the interest of theoretically oriented investigation toward the rigorous foundation of HHT. Finally, we will discuss the meaning of IMF in light of this theorem.

2. Review of EMD

EMD as originally proposed is implemented through a sifting process that is summarized as follows:

- (1) For any given data, $x(t)$, we identify all the local extrema.
- (2) Separately connect all the maxima and minima with natural cubic spline lines to form the upper, $u(t)$, and lower, $l(t)$, envelopes.
- (3) Find the mean of the envelopes as $m(t) = [u(t) + l(t)]/2$.
- (4) Take the difference between the data and the mean as the proto-IMF, $h(t) = x(t) - m(t)$.
- (5) Check the proto-IMF against the definition of IMF and the stoppage criterion to determine if it is an IMF.
- (6) If the proto-IMF does not satisfy the definition, repeat step 1 to 5 on $h(t)$ as many time as needed till it satisfies the definition.
- (7) If the proto-IMF does satisfy the definition, assign the proto-IMF as an IMF component, $c(t)$.
- (8) Repeat the operation step 1 to 7 on the residue, $r(t) = x(t) - c(t)$, as the data.
- (9) The operation ends when the residue contains no more than one extremum.

The flow chart of this sifting process is given in Fig. 1. Mathematically, the operation is given as follows:

$$\begin{aligned}
 x(t) - m_{1,1}(t) &= h_{1,1}(t); \\
 h_{1,1}(t) - m_{1,2}(t) &= h_{1,2}(t); \\
 &\dots \\
 &\dots \\
 h_{1,k-1}(t) - m_{1,k}(t) &= h_{1,k}(t); \\
 \Rightarrow h_{1,k}(t) &= c_1(t),
 \end{aligned}
 \tag{1}$$

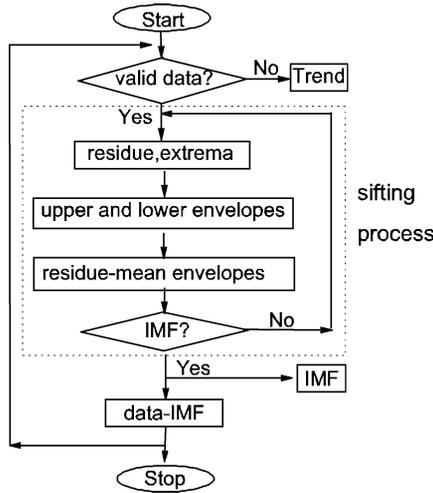


Fig. 1. The flow chart of the decomposition process of EMD.

in which the indices indicate the iteration of the same step. From this operation, we can see that

$$\begin{aligned}
 x(t) - m_{1,1}(t) &= h_{1,1}(t); \\
 h_{1,2}(t) &= h_{1,1}(t) - m_{1,2}(t) = x(t) - (m_{1,1} + m_{1,2}); \\
 &\dots \\
 &\dots \\
 h_{1,k}(t) &= h_{1,k-1}(t) - m_{1,k}(t) = x(t) - (m_{1,1} + m_{1,2} + \dots + m_{1,k}); \\
 &\Rightarrow c_1(t) = x(t) - (m_{1,1} + m_{1,2} + \dots + m_{1,k}).
 \end{aligned} \tag{2}$$

This is the step to extract the first IMF component. Subsequently, we have

$$\begin{aligned}
 x(t) - c_1(t) &= r_1(t); \\
 r_1(t) - c_2(t) &= r_2(t); \\
 &\dots \\
 &\dots \\
 r_{n-1}(t) - c_n(t) &= r_n(t); \\
 \Rightarrow x(t) - \sum_{j=1}^n c_j(t) &= r_n(t).
 \end{aligned} \tag{3}$$

Therefore,

$$r_1(t) = \sum_{j=1}^{k_1} m_{1,j} \tag{4}$$

From Eq. (3), we have

$$c_2(t) = r_1 - r_2 = \sum_{j=1}^{k_1} m_{1,j} - \sum_{j=1}^{k_2} m_{2,j} \tag{5}$$

Similarly, we should have

$$c_i(t) = r_{i-1} - r_i = \sum_{j=1}^{k_1} m_{i-1,j} - \sum_{j=1}^{k_2} m_{i,j} \tag{6}$$

Thus, all the IMF components, other than the first one, are the sums of spline functions. With his form, the basis formed in terms of IMFs through EMD can be shown to be complete automatically, for

$$\begin{aligned} x(t) &= \sum_{j=1}^n c_j + r_n \\ &= x(t) - \sum_{j=1}^{k_1} m_{1,j} + \left(\sum_{j=1}^{k_1} m_{1,j} - \sum_{j=1}^{k_2} m_{2,j} \right) \\ &\quad + \cdots + \left(\sum_{j=1}^{k_{n-1}} m_{n-1,j} - \sum_{j=1}^{k_n} m_{n,j} \right) + \sum_{j=1}^{k_n} m_{n,j} \\ &= x(t). \end{aligned} \tag{7}$$

From these operations, we can see that IMFs could sum up to the original signal precisely. The constituting spline function sums could cancel each other exactly. The whole EMD analysis as implemented here is built on spline function. Up to now, the EMD procedures seem to be simple and straightforward, except that there are two crucial elements in the EMD analysis need to be discussed and clarified: the definition of IMF against which the proto-IMFs have to compare with, and also the stoppage criterion according to which the operation could find a closure.

Now let us turn to the two critical elements in the EMD algorithm. We will first review the definition of IMF as given originally is this: Any function having the same numbers (or at most differing by one) of zero-crossings and extrema, and also having symmetric envelopes defined by local maxima and minima respectively is an Intrinsic Mode Function.

Second, let us discuss the stoppage criterion. Ever since the introduction of EMD, many different stoppage criteria were proposed. By judiciously select the stoppage criteria, EMD had produced informative and useful results. All the presently available stoppage criteria can be classified as follows:

(a) The Cauchy type criterion

The Cauchy type criterion was originally proposed by Huang *et al.* [1998]. Specifically, the sifting process will stop when the difference *SD* defined as is smaller than a preset value.

$$SD = \sum_{t=0}^T \frac{[h_{k-1}(t) - h_k(t)]^2}{h_{k-1}^2(t)} \tag{8}$$

According to this definition, the convergence condition could be stated mathematically as follows: For any given small number, ϵ , there exists a large number M of

iterations such that $SD < \varepsilon$ whenever the iteration of sifting number, K , is larger than M . The convergence is always satisfied empirically, but rigorous prove is still lacking. The above criterion is a global property as it is an integrated criterion over the whole data domain. Furthermore, the SD value so defined is heavily influenced by small proto-IMF values at particular locations. To make the criterion more stable, a variation is defined as

$$SD = \frac{\sum_{t=0}^T [h_{k-1}(t) - h_k(t)]^2}{\sum_{t=0}^T h_{k-1}^2(t)} \quad (9)$$

Though this definition is more stable, it is still global and even smoother. Still another variation along this line is to have SD defined as to be small everywhere.

$$SD = \frac{[h_{k-1}(t) - h_k(t)]^2}{h_{k-1}^2(t)} \quad (10)$$

Though this definition is more local, it is also influenced by the small local small value of the proto-IMF. All those above definition and variations, however, suffer from the fact that the definition cannot guarantee the results are indeed IMFs, for satisfying the criterion has nothing to do with either the numbers of extrema and zero-crossings or the symmetry of the envelope. To remedy this situation, there are two more new approaches.

(b) The mean value criterion

This one was proposed by Flandrin [2004], in which the SD defined as single term in Eq. (6)

$$SD = m_{i,k}(t) \quad (11)$$

The sifting will stop when SD is smaller than a pre-signed value everywhere. This definition is certainly better than the Cauchy condition, for it is related to the definition of IMF by requiring the mean of envelopes to be small. As it stands, it would force the envelopes to be symmetric; therefore, it satisfies one of the two critical characteristics of IMF.

(c) The S -number criterion

This form was proposed by Huang *et al.* [2003], which is related to another aspect of the definition of IMF. To implement this definition, one needs to count the number of extrema and zero-crossings. The S -number is defined as the number of consecutive sifting iterations in which the number of zero-crossings and extrema stay the same and are equal or differ by one.

Each of the above methods satisfies only one aspect of the IMF definition. All of them were based on a fussy approximation assumption, fully realized that over sifting will drain the physical meaning in the IMF components severely, and under

sifting would not produce satisfactory IMFs. All the above criteria give the users a wrong impression that the requirements should be set at a more stringent level with smaller SD value and larger S -numbers. Huang *et al.* [2004], however, pointed out that the more stringent condition on S -number would not necessarily produce better results. In a more detailed study by Wu and Huang [2009] and Wu and Huang [2010] it was further pointed out that by slightly changing the criterion, different number of IMFs could be generated. It makes the comparison results from different implementations of EMD difficult.

(d) The fixed sifting time criterion

In separate studies, Flandrin [2004], and Wu and Huang [2004] established that EMD is in fact a bank of dyadic filters. In a more recent study through systematic empirical trials by Wu and Huang [2010], they reached the conclusion that the dyadic property is valid only if one kept the iterations of sifting process around 10 times. The dyadic property would break down if the iteration number is too high or too low. The asymptotic state of infinite number of iterations in sifting would produce a result of frequency modulate (FM) waves with constant amplitude, almost approaching the result of the Fourier decomposition.

As we can see, none of the stoppage criteria is totally satisfactory, but all of them would serve the purpose if applied judiciously. The problem is that there is no rigorous mathematic standard for us to make decision. The determination of a stopping criterion to produce physically meaning IMFs is still a challenging objective to be reached in the implementation of EMD. At the present time, all the criteria will have to be implemented with some degree of fussiness and to satisfy the definition of IMF only approximately. Indeed, the fussiness in the stoppage criteria arises from a conflict between IMF definition and the presently used EMD implementation algorithm. This would be the subject of the next section.

3. The Conflict between IMF Definition and EMD Implementation Algorithms

As the above review indicates, the implementation of EMD has been based on the cubic natural spline for its flexibility and simplicity. At the same time, it has been clear understood that the definition of IMF could not be rigorously realized: for large number of siftings would produce an IMF better adhesive to the IMF definition, but such a function would not be physically meaningful. The optimal number of sifting iterations is illusive. As a result, all the stoppage criteria are somewhat fussy. This dilemma makes putting the EMD on a rigorous mathematical foundation extremely difficult, if possible at all. In the following we will prove that the dilemma arises from a conflict between definition of IMF and the algorithm based on spline fitting. As a result, the fussiness would be a feature we have to live with unless the spline implementation is modified. Let us start with the cubic natural spline algorithm as EMD is currently implemented. We will prove the following theorem that is the

core of the conflict:

For any function with sparsely populated extrema, the envelopes $u(t)$ and $l(t)$, constructed according to the EMD sifting procedures, have to degenerate to a pair of symmetric straight lines.

The proof will consist of two steps: The first step is to establish that envelopes will have to be a unitary cubic spline, and the second step is to establish the unitary spline line will degenerate to a straight line. The first step is given as follows: Assuming that we have an IMF, $c(t)$, extracted from a signal, $x(t)$, through the envelopes, $u(t)$ and $l(t)$, as the upper and lower envelopes and the mean $m(t)$ as given in Eq. (2). All these functions are piecewise cubic spline curves. If $c(t)$ satisfies the definition of IMF strictly, we should have

$$u(t) = -l(t). \tag{12}$$

Without loss of generality, if we also assume that the extrema of IMF are sparsely distributed with respect to the data points, i.e., there are at least two data points between a maximum and its neighboring minimum. The situation would be the case shown in Fig. 2, where t_1, t_3 , and t_5 are the locations of the maxima of $c(t)$, t_2 and t_4 the locations of minima; $u_1(t)$ and $u_2(t)$ are segments of $u(t)$, $l_0(t)$ the segment of $l(t)$. Then

$$\begin{aligned} u_1(t) &= A_1t^3 + A_2t^2 + A_3t + A_4 \quad t \in [t_1, t_3], \\ u_2(t) &= B_1t^3 + B_2t^2 + B_3t + B_4 \quad t \in [t_3, t_5], \\ l_0(t) &= C_1t^3 + C_2t^2 + C_3t + C_4 \quad t \in [t_2, t_4] \end{aligned} \tag{13}$$

Since there are at least two data points in $[t_2, t_3]$, equation $u_1(t) + l_0(t) = 0$ has at least four roots in $t \in [t_2, t_3]$. Notice that $u_1(t)$ and $l_0(t)$ are cubic functions, the

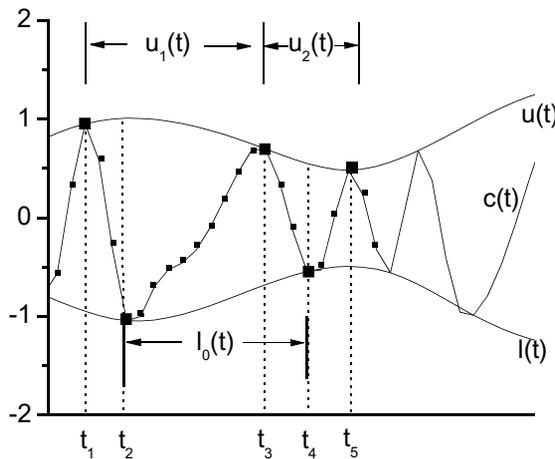


Fig. 2. Illustration of the upper and lower envelopes of an IMF and some extrema on them.

above Eq. (13) shows that they must have the same coefficients in their expressions, i.e., $A_i = -C_i$, $i = 1, 2, 3, 4$. Similarly, over the interval $t \in [t_3, t_4]$ we should have $B_i = -C_i$, $i = 1, 2, 3, 4$. Thus the two segmental cubic curves, $u_1(t)$ and $u_2(t)$, are actually two parts of the same cubic function. This conclusion is same with all the other segmental functions in other intervals. We finally found that the upper (lower) envelope of an IMF is in fact a unitary cubic function, although it is composed of a number of segmental cubic functions. In other word, all the maxima (minima) of $c(t)$ are on one single cubic curve,

$$u(t) = U_1 t^3 + U_2 t^2 + U_3 t + U_4 \quad (14)$$

Thus we finished the first step. Next, let a and b be the initial and end time points. As $u(t)$ is the upper envelope of $c(t)$, and also as $u(t)$ is a cubic natural spline as suggested by Huang *et al.* [1998], we should have the zero curvature at the ends:

$$u''(a) = u''(b) = 0. \quad (15)$$

This condition forces U_1 and U_2 to be zero identically. Therefore, $u(t)$ has to degenerate to a straight line. This finished the proof of the theorem.

An important corollary of the above theorem is this:

Other than data with strictly monotonic varying amplitudes, the slope of the final straight line for general data has to be zero.

It is easy to see that the only cases that the slope is not zero would be for strictly monotonic increasing or decreasing in amplitude of the oscillating data. As most general data are not strictly monotonic in oscillating amplitude, the slope of the straight line has to be zero.

This eventual zero slope straight line envelope result agrees exactly with the asymptotic state in Wu and Huang [2010] empirically observed result that iteration of siftings *ad infinitum* will produce IMF of constant amplitude. Additionally, if one takes the mirror extension to treat the end effects by supply the extrema at boundaries to extend the envelopes when implementing EMD, the amplitude of the envelope will always be constant.

In EMD decomposing procedure, we extract the component with the highest frequency from the data and proceed to the component with the second highest frequency in the residual data. The procedure continues until the trend function is obtained. With the implementation of EEMD, the extrema distribution of the second or higher modes would meet the sparse condition we set above. Accordingly, the amplitude of IMF2 or higher would have to be constant, if the definition of the IMF is strictly satisfied.

If spline function of higher order is used, the above straight line limitation will still be valid only the sparse condition will be satisfied at a higher IMF components. Higher order spline, however, is in fact impractical and could even result in non-convergence and also produce artificial higher frequency variations. Envelope defined with function other than the natural spline zero curvature condition

would require other hardness to define artificial conditions. Therefore, the above conclusion could be extended to any spline implementation of EMD with any order.

4. Discussion and Conclusion

Now, we have two results, one empirical, in the form of Wu and Huang [2010], and one theoretical, in the form of the present theorem, to highlight the conflict of EMD and its eventual goal: to produce physical meaningful IMFs. Both the empirical and the theoretical results point to the asymptotic state of infinite number of sifting iterations. The empirical result further established the processes how the final IMFs of constant amplitude are reached. These two results together help us to understand an oddity of the sifting operation: the ability of sifting operation to split a perfect IMF further. By understanding the processes, we could gain further understanding of the meaning of IMFs and the operation in obtaining them.

The interesting oddity of the sifting operation is its ability of splitting a function that seems to be a perfect IMF into more separate components. Consider the following trigonometric identity:

$$2 \sin(\alpha + \beta) \cdot \sin(\alpha - \beta) = \cos 2\alpha - \cos 2\beta \quad (16)$$

This function could be considered as consisted of a carrier and a long period envelope based on the left hand side expression. In this form, the function clearly satisfies the definition of IMF and should be regarded as one. At the same time, it could also be considered as two co-existing sinusoidal function each have a constant frequency based on the right hand side expression. For α and β having value close to each other, the dyadic filter bank should never be able to separate them. The fact that it is possible for EMD to split the function into separate components at all as shown in Huang *et al.* (1998) was surprising and puzzling. The puzzle is this: Why EMD could decompose a perfect IMF? With the present theorem and the empirical observations by Wu and Huang [2010] we can explain this oddity as a logical consequence of EMD. The reason is that the envelope defined by EMD is only an approximation of the perfect IMF in terms of the cubic spline function. As the perfect envelope is not a cubic nature spline function, EMD will force the envelope to be one. In the processes, the sifting iteration number will have to be higher and higher, and the filter bank will have narrower and narrower bands. Thus, the ability to separate components increases as well. Unfortunately, with increasing number of iterations, the envelope will approach a straight line. As the increasing iterations makes the filter bank central frequency getting closer; thus enable the EMD operation to split the beating waves, given on the left hand side, into the individual components given on the right hand side.

Let us illustrate this condition with the following example. Consider the data given by

$$x(t) = \sin(\pi t/64) \cdot \sin(\pi t/256), \quad \text{for } t = 0 : 1024 \quad (17)$$

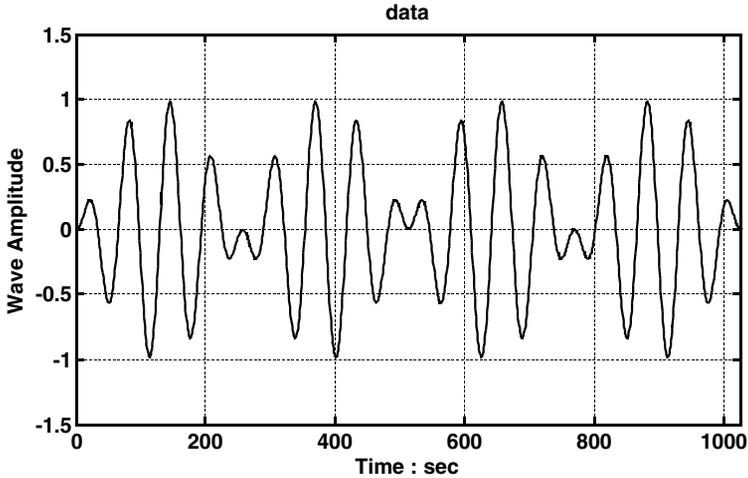


Fig. 3. The data of a modulated wave train.

The wave form is given in Fig. 3. The wave is definitely a carrier modulated by a low frequency envelope. When we subject this data for cubic natural spline based EMD, we got the following IMFs shown in Fig. 4, depending on different iteration numbers.

Figure 4(a) shows the result from a low iteration number result at 10 as suggested by Wu *et al.* [2009]; here we got 6 IMFs, but we show only 5 here. The modulated wave pattern is the dominate feature, but we can see some leakage already to the second IMF component. Figure 4(b) shows the result from a intermediate iteration number result at 100; here we got 9 IMFs, but we show only the first 5 here for clarity of the most important components. The modulated pattern is fading away, but still visible in the first IMF component. The second independent component is growing to a comparable magnitude with the first now, because of the decreasing frequency difference and inevitable leakage. Of course, one could also view this as increasing power of filtering, albeit in the Fourier sense only. The filter bank has much closer peak frequency due to the higher iteration number as found by Wu and Huang (2010). Figure 4(c) shows the result from a high iteration number result at 10,000; here we got 28 IMFs, but again we show only the first 5 here. The modulated wave pattern is totally invisible. The two dominate IMFs as given in IMF1 and IMF2 are of comparable magnitude all with constant amplitudes. As the centroid frequencies are getting closer and the leakage also becoming increasingly serious as indicated by the non-zero components other than the first two IMFs. At this stage, the envelope had approached its constant zero slope form as predicted by the theorem presented above.

Large number of iterations indeed could be considered as a filter bank having extremely sharp filters. Over zealous sifting with a large number of iteration has shown to be able to achieve a rudimentary separation as in Huang *et al.* (1998).

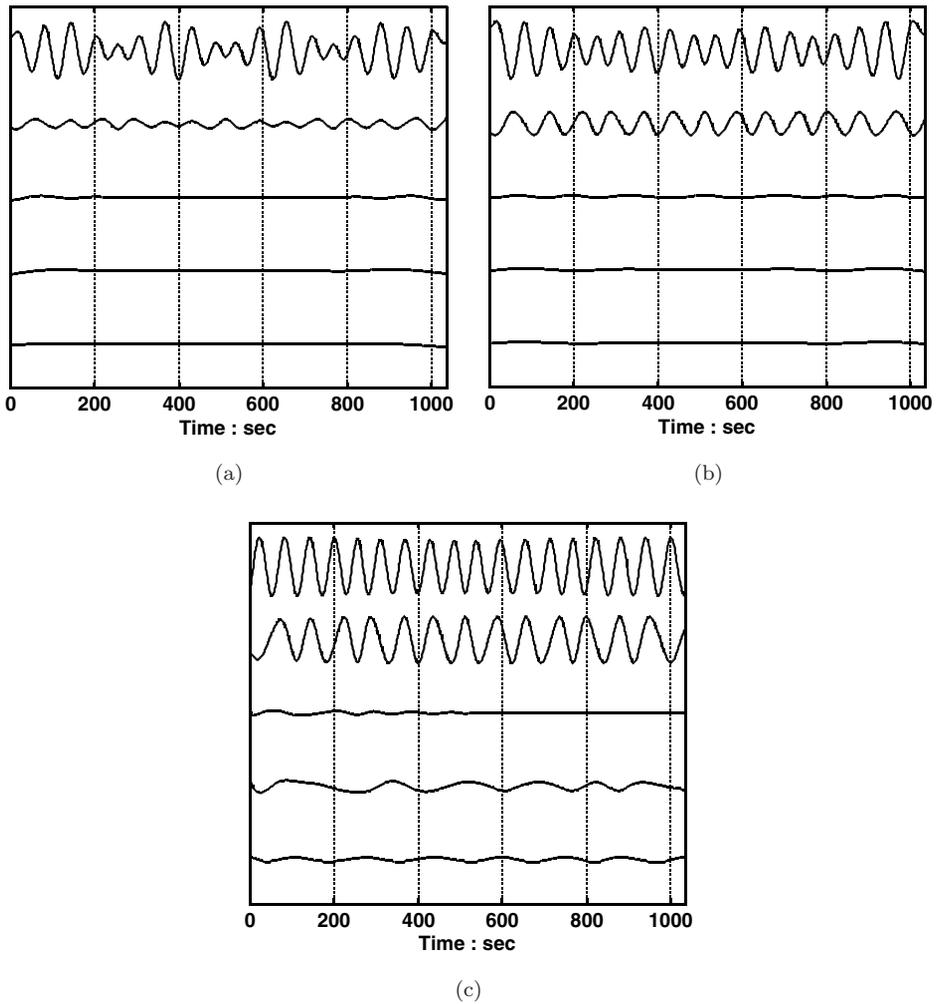


Fig. 4. The IMFs from different iteration numbers in the EMD implementation. (a) for 5 iterations, the modulation pattern is the dominate feature. (b) for 100 iterations, the modulation pattern is still visible. (c) for 10,000 iterations, the modulation wave pattern fades away totally and two individual components are clearly presented here.

Now, let us examine the mechanism how the EMD separate the perfect IMF as given by Eq. (17). Figure 5 gives the data, the true and spline envelopes. From this figure, one can see that the cubic natural spline indeed gives a seemingly acceptable envelope as compared to the original theoretical one. Because the spline envelope is an approximation, there are visible differences. To show the difference more clearly, we plotted the difference between the true envelope and the upper and lower envelopes determined by cubic spline in Fig. 6, together with the mean of the spline envelopes; the difference is sizeable and could have sharp local changes. The mean is much smoother. Such difference provides the mechanism that the EMD based

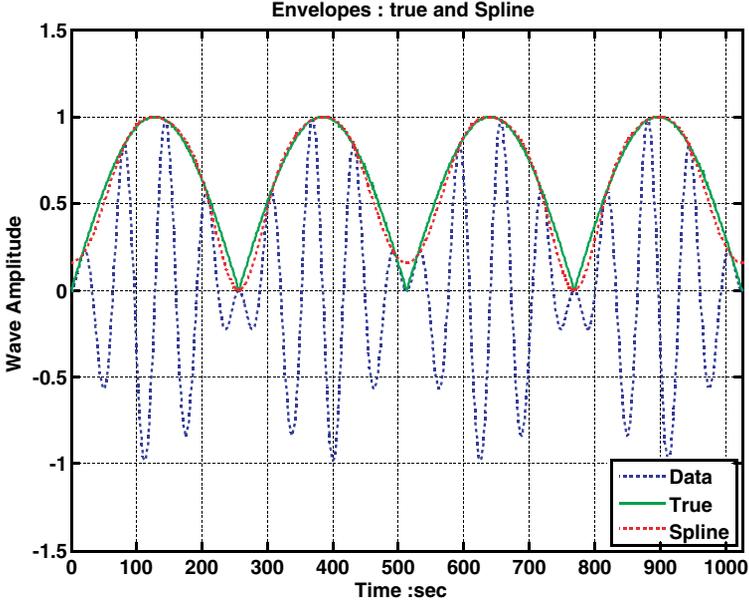


Fig. 5. Data of a modulated wave train with true and spline envelopes. The difference between the true and spline envelopes is clearly visible.

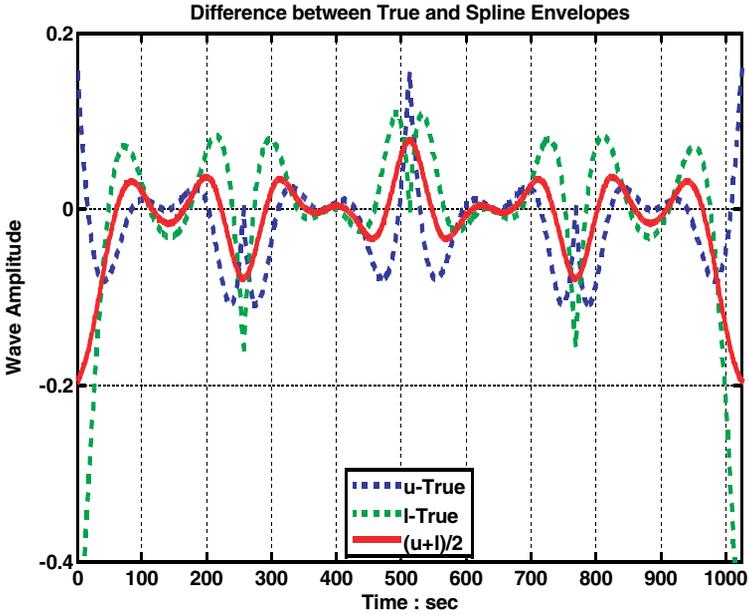


Fig. 6. The difference between true and spline envelope for a data with unit amplitude.

on cubic natural spline gradually split the modulated wave to separate component. As any modulated wave could be considered as beating of separated components, where should we draw the line of physical meaningfulness? It is a philosophical as well as a practical problem. Our position is consistently that modulated wave is mathematically equivalent to its separate components. However, to separate the beating patterns into their constant amplitude individual components would be equivalent to revert to a Fourier view of data. Unfortunately, for the Fourier view to be legitimate, the processes would have to be both linear and stationary, when the constant amplitude representation would be perfect. For natural data that could be both nonlinear and nonstationary, we should keep the natural modulation wave forms as pristine and truthful as possible. We, however, have a problem: what are the true envelopes of these modulated wave patterns? And how to get them? We proposed spline as the basis for the approximations; they should be treated as any other basis, but it is adaptive. Knowing that EMD is an approximation, and that the error will build up with each sifting iteration, we should keep the number of iterations low. Therefore, over sifting should be avoided, and such practice is not recommended under any situation. With the help of this theorem and the empirical observations, we finally resolve the puzzle how the EMD could separate a perfect modulated wave into its individual components: all because the cubic natural spline envelope is an approximation.

EMD and many of its variations such as EEMD and CEEMD have proved to be eminently useful. There is nothing wrong with the definition of the IMF; the problem arises from the implementation of EMD. For lack of any viable alternative, spline fitting was proposed. The problem associated with it has been identified early in the development of the EMD method, which had forced the empirical implementation to obtain the approximate IMFs that is physically meaningful. The annoying problem of lacking an objective set of its implementation rules arises from the insistence of being physically meaningful, which could not be defined rigorously and mathematically. Having established the limiting straight line form of IMF amplitude under the rigorous symmetry requirement, we seem to have pinpointed the root of the theoretical conflict. Therefore, there is a new urgency to find a way to resolve it. There are two options available for us to overcome this difficulty.

The first option is to eschew spline approach. As the conflict here arises through the implementation of EMD by natural spline, avoid spline would eliminate the conflict. In fact, spline implementation for EMD is based on the convenience. The envelope does not have to be a spline function as the recent work by Hou *et al.* [2009]. Future development is still under way.

The second option is to accept an approximation for the IMFs, i.e., we relax the stoppage condition and IMF definition, if the EMD is implemented through spline as suggested by Huang *et al.* [1998]. This is the reason that the sifting condition should never be enforced to the limit of $SD = 0$, and also why Huang *et al.* [2003] and Wu and Huang [2010] advised against a large number of sifting. It was pointed out at the introduction of EMD that the effect of sifting is to get rid of riding waves,

but the side effect is to make the amplitude more uniform. Large number of siftings would make the amplitude of IMF approaching to a constant as consequence of sifting and also of the boundary condition of the natural spline function and mirror extension on the boundary. Either way, the physical meaning would be drained off the resulting IMF. Such practice should be avoided at all the times.

With the above observations, we can see that rigorous adhesive to the IMF definition with spline implement would lead to a ridiculous result as clearly stated by the previous studies by Huang *et al.* [2003] and Wu and Huang [2010]. But the approximate solution suggested by Huang *et al.* [1998] could lead to the annoying situation: different stoppagecriterion would lead to different resulting IMF sets. In other words, we need an objective standard for the approximation. The difficulty can be attributed to a lack of an objectively determined optimal stoppage criterion, which is an illusive goal so far. Although EEMD had greatly reduced the mode mixing disparity considerably, it has not fully resolved the dilemma. Unfortunately, exhausted search had failed to discover a solution. Many of the optimization schemes of the stoppage criterion test as a function of iteration number, n , end up in compact convex sets with the solution at the unacceptable $n = \infty$. Among the few available options, the most attractive one is the fixed number of iterations in sifting. Wu and Huang [2009] had shown that if the iteration number is set at 10, the EMD would be a bank of dyadic filters, the shortcomings notwithstanding. This selection is defensible based on another consideration: the separation of IMF components and the minimum of leakage. As shown by Wu and Huang [2004] and Flandrin [2004], even when the EMD is optimum as a bank of dyadic filters, there are still leakages between neighboring components. The leakage would increase with the ratio of mean frequencies from the neighboring component dropping to less than 2 [Wu and Huang (2010)]. As the ratio decreasing the orthogonal property would be even worsen. To maintain maximum separation and minimum leakage, the dyadic filter bank is the optimal choice. Thus, if we choose the fixed iteration time of sifting, we could have an objective standard for EMD implementation. In this case, we should always have $\log_2 N$ IMF components from EEMD, with N as the total number of data points.

In this paper, we reported a theoretical result to explain some of the puzzling phenomena associated with EMD. The problem is clear now, but the conflict is still waiting for its full resolution. It should be realized that the EMD as implemented now yields an approximation with respect to the cubic spline basis. Under the present circumstance, we further conclude that a fixed low number of iterations would be the best option at this time, for it delivers the best approximation.

Acknowledgment

The work was supported by the National High Technology Research and Development Program of China (No. 2008AA09A402), Key Project of National Natural Science Foundation of China (No. 40730842), Open Fund of the Key Laboratory

of Ocean Circulation and Waves, CAS (KLOCAW0905), Basic Natural Research Found of the First Institute of Oceanography, SOA (GY02-2009G08), National Basic Research Program of China (No. 2007CB816002), National Science Foundation of China (No. 40776018), National Key Technology R&D Program (No. 2006BAB18B02), and Chinese Polar Science Strategy Foundation (No. 20070208). NEH was supported in part by a grant from Federal Highway Administration of USA, DTFH61-08-00028.

References

- Battista, B. M. *et al.* (2007). Application of the empirical mode decomposition and Hilbert-Huang transform to seismic reflection data. *Geophysics*, **72**(2): 29–37.
- Camp, J. B. *et al.* (2009). Searching for gravitational waves with the Hilbert-Huang Transform. *Adv. in Adap. Data Analy.*, **1**(4): 643–666.
- Chen, J. (2009). Application of empirical mode decomposition in structural health monitoring: Some experience. *Adv. in Adap. Data Analy.*, **1**(4): 601–621.
- Chen, S. H. *et al.* (2008). Improving Empirical Mode Decomposition using support vector machines for multifocus image fusion. *Sensors.*, **8**: 2500–2508.
- Fauchereau, N. *et al.* (2008). 2-D Empirical Mode Decomposition on the sphere, application to the spatial scales of surface temperature variations. *Hydrol. Earth Syst. Sci. Discuss.*, **5**: 405–435.
- Flandrin, P. Rilling, G. and Goncalves, P. (2004). Empirical mode decomposition as a filter bank. *Signal Processing Letters, IEEE*, **11**(2): 112–114.
- Hou, T. Y. Yan, M. P. and Wu, Z. (2009). A variant of the EMD method for multi-scale data. *Adv. in Adap. Data Analy.*, **1**(4): 483–516.
- Huang, N. E. and Wu, Z. (2008). A review on Hilbert-Huang transform: Method and its applications to geophysical studies. *Rev. Geophys.*, **46**(2): RG2006.
- Huang, N. E. *et al.* (1998). The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis. *Proc. R. Soc. Lond. A*, **454**: 903–995.
- Huang, N. E. *et al.* (1999). A new view of nonlinear water waves—The Hilbert spectrum. *Annu. Rev. Fluid Mech.*, **31**: 417–457.
- Huang, N. E. *et al.* (2003). A confidence limit for the empirical mode decomposition and Hilbert spectral analysis. *Proc. R. Soc. Lond. A*, **459**: 3217–2345.
- Huang, N. E. *et al.* (2009). Reductions of noise and uncertainty in annual global surface temperature anomaly data. *Adv. in Adap. Data Analy.*, **1**(3): 447–460.
- Iyengar, R. N. and Kanth, S. T. G. R. (2006). Forecasting of seasonal monsoon rainfall at subdivision level. *Current Science*, **91**(3): 350–356.
- Khalidi, K. Alouane, M. T. and Boudraa, A. (2010). Voiced speech enhancement based on adaptive filtering of selected intrinsic mode function. *Adv. in Adap. Data Analy.*, **2**(1): 65–80.
- Liang, H. L. *et al.* (2005). Empirical mode decomposition: A method for analyzing neural data. *Neurocomputing*, **65–66**: 801–807.
- Linderhed, A. (2009). Image empirical mode decomposition: A new tool for image processing. *Adv. in Adap. Data Analy.*, **1**(2): 265–294.
- Liu, Z. *et al.* (2007). Bidimensional Empirical Mode Decomposition for the fusion of multispectral and panchromatic images. *International J. Remote Sens.*, **28**(18): 4081–4093.

- Molla, M. K. I. Sumi, A. and Rahman, M. S. (2007). Analysis of temperature change under global warming impact using Empirical Mode Decomposition. *International J. Information Technology*, **3**(2): 131–139.
- Nunes, J. C. and Deléchelle, E. (2009). Empirical mode decomposition: Applications on signal and image processing. *Adv. in Adap. Data Analy.*, **1**(1): 125–175.
- Nunes, J. C. *et al.* (2005). Texture analysis based on local analysis of the bidimensional Empirical Mode Decomposition. *Mach. Vision Appl.*, **16**(3): 177–188.
- Qian, C. *et al.* (2009). On the secular change of spring onset in Stockholm, *Geophysical Research Letter*, **36**: L12706, doi:10.1029/2009GL038617.
- Ruzmaikin, A. and Feyman, J. (2009). Search for climate trends in satellite data. *Adv. in Adap. Data Analy.*, **1**(4): 667–679.
- Shi, W. *et al.* (2009). A two-dimensional empirical mode decomposition method with application for fusing panchromatic and multispectral satellite images. *International J. Remote Sens.*, **30**(10): 2637–2652.
- Sinclair, S. and Pegram, G. G. S. (2005). Empirical Mode Decomposition in 2-D space and time: A tool for space-time rain fall analysis and nowcasting. *Hydrology and Earth System Sciences*, **2**: 289–318.
- Wilson, M. F. J. *et al.* (2007). Multiscale terrain analysis of multibeam bathymetry data for habitat mapping on the continental slope. *Mar. Geodesy.*, **30**: 3–35.
- Wu, Z. and Huang, N. E. (2004). A study of the characteristics of white noise using the empirical mode decomposition method. *Proc. R. Soc. Lond. A*, **460**: 1597–1611.
- Wu, Z. and Huang, N. E. (2009). Ensemble Empirical Mode Decomposition: A noise assisted data analysis method. *Adv. in Adap. Data Analy.*, **1**(1): 1–41.
- Wu, Z. Huang, N. E. and Chen, X. (2009). The multi-dimensional ensemble empirical mode decomposition method. *Adv. in Adap. Data Analy.*, **1**(3): 339–372.
- Wu, Z. and Huang, N. E. (2010). On the relations of empirical mode decomposition to the Fourier transform and wavelet analysis. Submitted to *Appl. Comput. Harmon. Anal.*
- Wu, Z. Schneider, E. K. Kirtman, B. P. Sarachik, E. S. Huang, N. E. and Tucker, C. J. (2008). The modulated annual cycle: An alternative reference frame for climate anomalies. *Climate Dynamics*, **31**: 823–841. DOI: 10.1007/s00382-008-0437-z.
- Yeh, J. R. Shieh, J. S. and Huang, N. E. (2010). Complementary ensemble empirical mode decomposition: A novel noise enhanced data analysis method. *Adv. in Adap. Data Analy.*, **2**(2): 135–156.
- Zhang, R. R. *et al.* (2003). Hilbert-Huang Transform analysis of dynamic and earthquake motion recordings. *J. Eng. Mech.*, **129**(8): 861–875.