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BOUNDARY EXTENSION AND STOP CRITERIA FOR EMPIRICAL MODE DECOMPOSITION

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In this paper, a new idea about the boundary extension has been introduced and applied to the Empirical Mode Decomposition (EMD) algorithm. Instead of the traditional mirror extension on the boundary, we propose a ratio extension on the boundary. We also adopt the stop criteria by Rilling *et al.* for B-Spline based EMD algorithm. Numerical experiments are used for empirically assessing performance of the modified EMD algorithm. The examples indicate that the ratio boundary extension indeed improves the result of the original EMD.

Keywords: EMD; boundary extension; stop criteria.

1. Introduction

Signal analysis is an important tool in both pure research and practical applications. Traditional data analysis methods such as Fourier analysis, based on the linear stationary assumption have been shown to be efficient for processing of linear and stationary data. However, these methods are less suitable for analyzing nonlinear and nonstationary data. In 1998, Huang *et al.* (1989) presented a new data analysis method, the Empirical Mode Decomposition (EMD) for analyzing nonlinear and nonstationary data. The EMD decomposes any signal $s(t)$ into a finite number of intrinsic mode functions (IMFs)

$$s(t) = \sum_{j=1}^{M} \varphi_j(t), \quad \text{where } \varphi_j(t) = a_j(t)\cos\theta_j(t).$$

Over the past decade, the EMD has gained more and more recognition. Most of the progress has been in its application. The underlying mathematical problems have been mostly left untreated. Huang listed seven outstanding mathematical problems of EMD in his book [Huang (1989)]. One of the outstanding problems is the data prediction problem for nonstationary processes (end effects). Since we are dealing with finite data, the algorithms must be adjusted to use some form of boundary conditions. For the EMD, the end points are problems again. And the influence of the ends will propagate into the data range during the sifting procedure. The extension of data, or data prediction, is a risky procedure even for linear and stationary processes. It is much harder for the nonlinear and nonstationary processes. Huang (1989) mentioned that all that is needed to be predicted for EMD are only the values and locations of the next several extrema, not all the extended data. Such a limited goal notwithstanding, the task is still challenging. The traditional way to extend the data beyond the existing range for EMD is symmetric extension around the boundary.

In this paper, a new method of the boundary extension is proposed. Instead of the traditional symmetric extension on the boundary, we suggest that the boundary should be extended based on the trend of the signal. Namely, we predict the signal based on the pattern of the signal. Although the signals we are dealing with are nonlinear and nonstationary, the distance between extreme points still will indicate how the frequency changes. Based on this fact, we will use the ratio of the distance between the nearest successive extreme points on the boundary to predict the location of added extremal points. Also, since the signal is amplitude modulated, instead of supposing the amplitude of the following extremal points is the same as its mirror extrema, we will use quadratic interpolation to determine the value at the added extremal points. Experimental results show that this novel idea does work better than the original EMD algorithm.

This paper is organized as follows. In Sec. 2, we describe the current EMD algorithm; in Sec. 3, we present the details of the ratio boundary extension; in Sec. 4, we discuss the stop criteria of the EMD algorithm; in Sec. 5, we compare the numerical results of the original EMD algorithm and the new boundary extension algorithm; some conclusions are made in Sec. 6.

2. Basic EMD

The EMD, in contrast to most of the earlier methods, works in temporal space directly rather than in the corresponding frequency space; it is intuitive, direct, and adaptive, with a posterior defined basis derived from the data. The decomposition is based on a simple assumption that any data consists of different simple intrinsic modes of oscillation. Each component is defined as an intrinsic mode function (IMF) satisfying the following conditions [Huang *et al.* (1989)]:

(i) In the whole data set, the number of extrema and the number of zero crossings must be either equal or differ at most by one.
(ii) At any data point, the mean value of the envelope defined using the local maxima and the envelope defined using the local minima is zero.

With the above definition of an IMF, one can then decompose any function through a sifting process.

In order to extract the first IMF $c_1(t)$, the component with the highest frequency embedded in the original signal $s(t)$, we use the following sifting process:

Sifting procedure:

(1) Compute a mean envelope $m_1(t)$ of the signal $s(t)$.
(2) Let $h_1(t) = s(t) - m_1(t)$ be the residue.
(3) If $h_1(t)$ is an IMF, STOP; else, treat $h_1(t)$ (with its extrema) as a new signal to obtain $h_{1,1}(t)$.
(4) If $h_{1,1}(t)$ is an IMF, STOP; else, continue the same process

$$\begin{aligned} h_{1,1}(t) &= h_1(t) - m_{1,1}(t) \\ &\cdots \\ h_{1,k}(t) &= h_{1,k-1}(t) - m_{1,k}(t) \end{aligned}$$

Generally, after a finite number k_1 times, $h_{1,k_1}(t)$ will be an IMF.

Denote by $c_1(t)$ the first IMF. Set $r_1(t) = s(t) - c_1(t)$. And repeat the sifting procedure:

$$\begin{aligned} r_2(t) &= r_1(t) - c_2(t), \\ &\cdots \\ r_N(t) &= r_{N-1}(t) - c_N(t). \end{aligned}$$

End when r_N has at most one extrema.

Thus $s(t) = \sum_{j=1}^{N} c_j(t) + r_N(t)$ is decomposed into finitely many IMFs.

3. Boundary Extension for Nonlinear and Nonstationary Signals

The basic operation in the sifting procedure is the estimation of the mean envelope. There are two typical methods to calculate the mean envelope. One is using Cubic Spline Interpolation of the local maxima (respectively local minima) of $s(t)$ to get the upper envelope $U(t)$ (respectively lower envelope $L(t)$), then compute the average $m(t) = (U(t) + L(t))/2$ as the mean envelope. Another way is using the moving average of the extrema as a combination of B-Splines as proposed by Riemenschneider *et al.* (2005) and Chen *et al.* (2006), the mean is calculated as

$$m(t) = \sum \frac{1}{4}[s(\tau_{j+1}) + 2s(\tau_{j+2}) + s(\tau_{j+3})]B_{j,4,\tau}(t),$$

where τ_j are the local extrema point of $s(t)$.

Due to the finite observation lengths of the signal, we have to extend the extrema before we apply the Cubic Spline Interpolation to find the upper/lower envelope or use B-splines to find the mean envelope. The general method is to add extrema by mirror symmetry with respect to the end points or with respect to the extrema which are closest to the end points. Another approach was outlined by

Wu and Huang [2009]. In this section, we will focus our attention on the following two problems:

(1) How to predict the proper location of the extended extrema?
(2) How to predict the proper value of the extended extrema?

We will use the right hand end of the data to illustrate our idea. Since we are dealing with nonlinear and nonstationary signals, the frequency of the signal (i.e. the distance between two extrema) will change with time. If we use symmetric extension with respect to the last extrema, the distance between the two extended extrema is the same as the distance between the two extrema which are closest to the end points. It works for a signal with constant frequency, but it fails when the frequency is not fixed. Let us look at the signal in Fig. 1. For this signal, we would expect that the frequency of the signal is decreasing, so the distance between two successive extreme points will become larger. Mirror extension will not predict the correct location of the added extremal points. Since the distance between the extreme reveal the frequency information, we can predict the location of the next extreme point based on the pattern of the current extremal points. Here is our strategy: we use linear approximation to estimate the change of the frequency of the signal at the end of the signal. Suppose the locations of the last three maximum points of the signal $s(t)$ are τ_{-3}, τ_{-2} and τ_{-1}. Let $r_{\max} = (\tau_{-1} - \tau_{-2})/(\tau_{-2} - \tau_{-3})$ and the location of the first extended maximum be τ_1. Then the distance between any two successive maxima should approximately keep the constant ratio $r_{\max}$, i.e. τ_1 should satisfy $(\tau_1 - \tau_{-1})/(\tau_{-1} - \tau_{-2}) = r_{\max}$. Similarly, suppose the locations of the last three minimum points of the signal $s(t)$ are η_{-3}, η_{-2} and η_{-1}. Let $r_{\min} = (\eta_{-1} - \eta_{-2})/(\eta_{-2} - \eta_{-3})$ and the location of the first extended minimum be η_1. Then η_1 should approximately satisfy $(\eta_1 - \eta_{-1})/(\eta_{-1} - \eta_{-2}) = r_{\min}$. To get

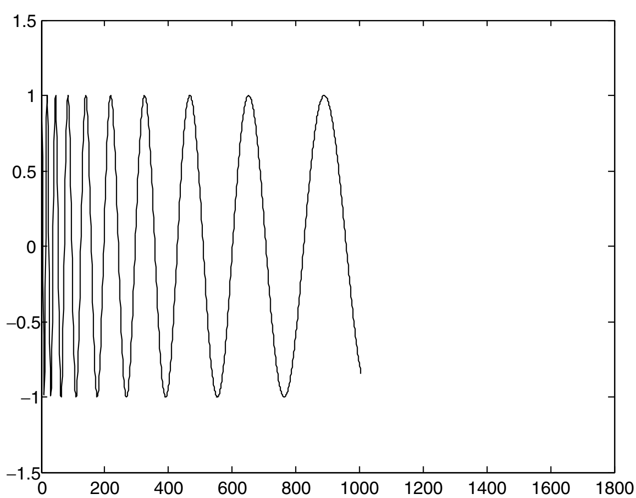
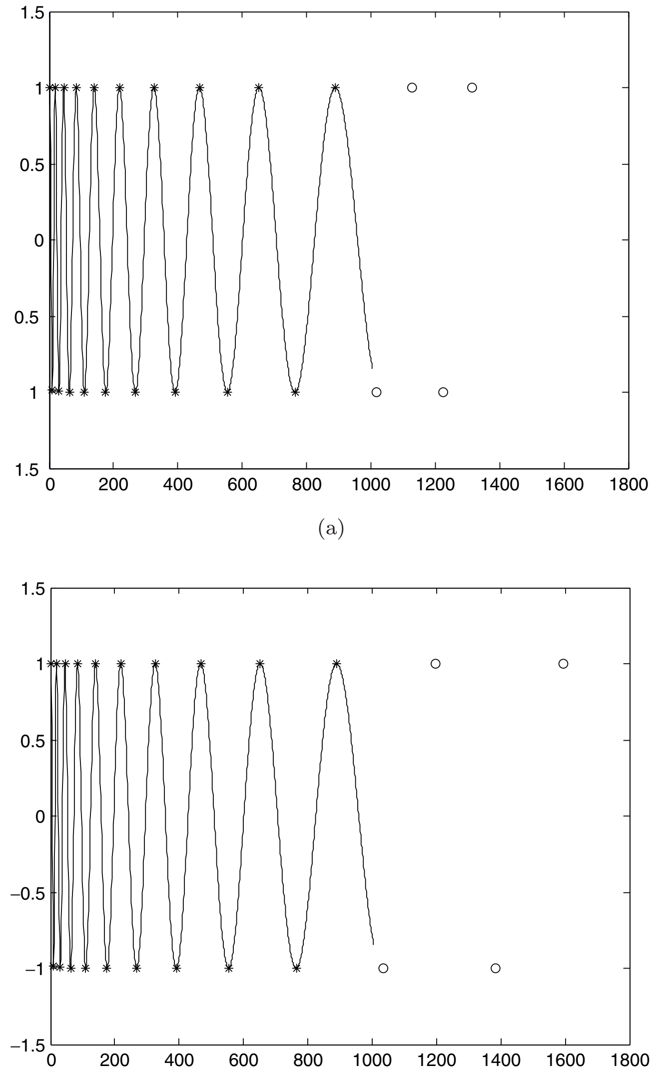


Fig. 1. A frequency modulated signal.



(b)

Fig. 2. The circle on the graph denote the extended extrema. (a) is with symmetry extension, (b) is with ratio extension.

better result, we take the mean ratio, $r = (r_{\max} + r_{\min})/2$, and find τ_1 and η_1 such that $(\tau_{-1} - \tau_{-2})/(\tau_{-2} - \tau_{-3}) = r$ and $(\eta_1 - \eta_{-1})/(\eta_{-1} - \eta_{-2}) = r$. Thus, we get the location of the two added maxima and two added minima (see Fig. 2(b)).

After we estimate the location of the extended extrema, we need to predict the value at the added extremal points. As mentioned in the previous section, the residual of the signal usually is not a constant. The residual shows the trend of the signal and usually it is monotonic or has one extreme. Let us look at the simplest case, one IMF + one monotonic trend as in Fig. 3. If we use symmetric extension

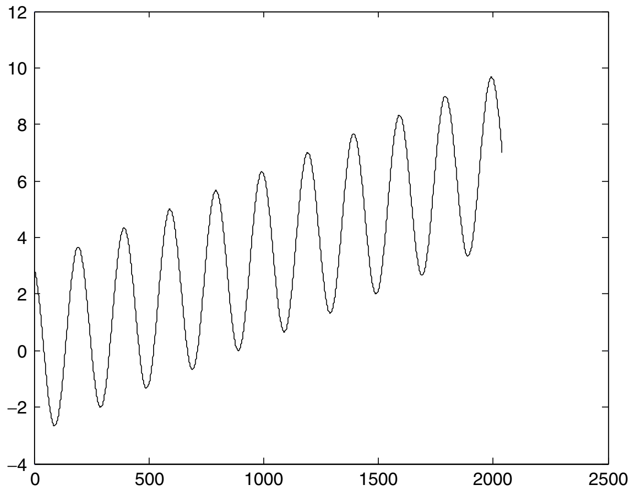


Fig. 3. A signal with one sinusoid and one straight line residue.

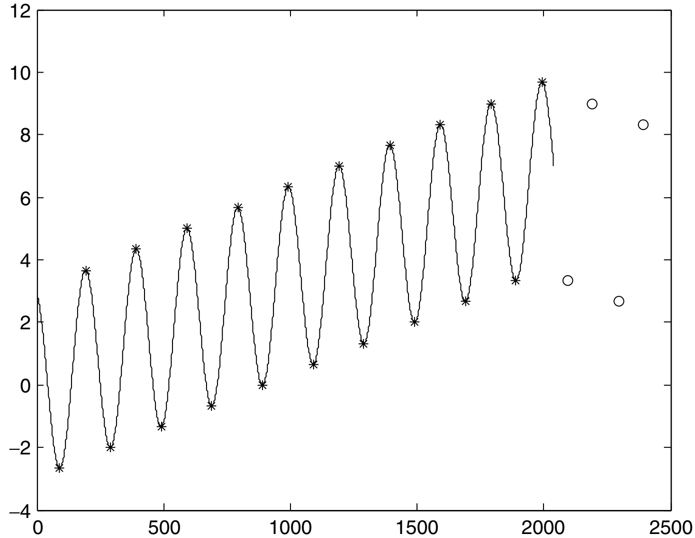
with respect to the last extreme, the extremal points will be extended as Fig. 4(a). Obviously, the data extension does not give the proper prediction of the original signal. It is naturally for us to think that the extrema of the signal should be extended as in Fig. 4(b).

Based on this fact, we propose to predict the value of extended maximum by quadratic interpolation on the last three maxima of the original signal.

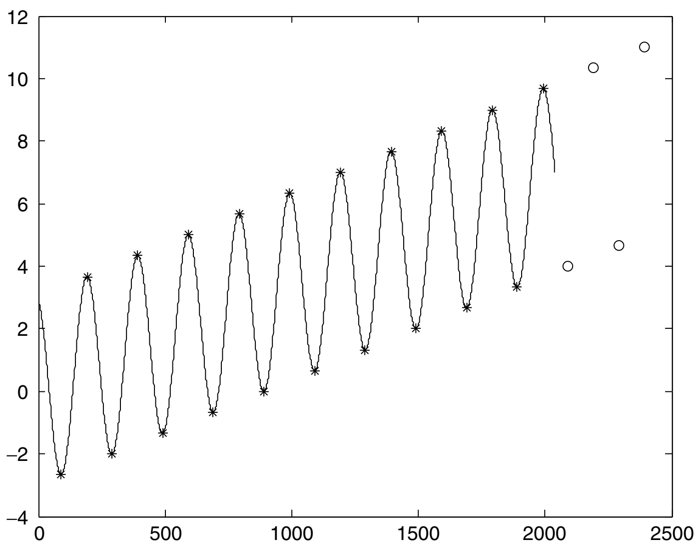
Combining the above mentioned ideas about predicting the location and value of the new maximum, we set up the following boundary extension algorithm:

Ratio Boundary Extension

(1) Find the locations of the last three maximum points, say τ_{-3}, τ_{-2} and τ_{-1}.
(2) Calculate the ratio of the distance between the last three maximum points, $r_{\max} = (\tau_{-1} - \tau_{-2})/(\tau_{-2} - \tau_{-3})$.
(3) Find the locations of the last three minimum points, say η_{-3}, η_{-2} and η_{-1}.
(4) Calculate the ratio of the distance between the last three minimum points, $r_{\min} = (\eta_{-1} - \eta_{-2})/(\eta_{-2} - \eta_{-3})$.
(5) Calculate the mean ratio $r = (r_{\max} + r_{\min})/2$.
(6) Find the location (say τ_1) of the first extended maximum points, such that $(\tau_1 - \tau_{-1})/(\tau_{-1} - \tau_{-2}) = r$
(7) Quadratic interpolation on the last three maximums of the given signal.
(8) Calculate the first extended maximum using the quadratic function.



(a)



(b)

Fig. 4. The circle on the graph denote the extended extrema. (a) is with symmetry extension, (b) is with quadratic extension.

The same idea is applied to estimate the location and values of the second extended maximum points and two extended minimum points. And we extend extrema to the other side of the signal by the same method. We call this boundary extension as ***Ratio Boundary Extension*** since the distance of the extrema and the value of the extrema are proportional to the existing extrema. In most of the cases, the ratio boundary extension works pretty well. But in some extreme situations, the ratio boundary extension may fail the maximum and minimum points interlacing condition. In that case, we will use the mirror extension about the first extreme point instead.

The results of ratio boundary extension of the signals in Figs. 3 and 1 are shown in Figs. 4(b) and 2(b), respectively.

4. Stop Criteria for Sifting

In Sec. 2, we gave the definition of an IMF. But it is difficult to use this definition directly to evaluate whether a signal is an IMF in the numerical implementation. Thus we need to set up some stop criteria to determine whether a signal could be viewed as an IMF during the sifting procedure. The choice of stopping criteria is very important to the application of EMD, sifting too many steps may lead to loss of amplitude variation and physical meaning.

The stopping condition imposed in Huang's paper [Huang *et al.* (1989)] is to limit the standard deviation computed from two consecutive results in the shifting process:

$$\mathrm{SD} = \frac{\sum_t |h_{i,k-1}(t) - h_{i,k}(t)|^2}{\sum_t h_{i,k-1}^2(t)}.$$

If SD is smaller than a predetermined value, the sifting process will be stopped. Wu and Huang (2009) discussed the disadvantage of the global standard deviation criteria and introduced the idea of local stoppage criteria.

Instead of using the standard deviation SD as the stop criteria, Rilling *et al.* (2003) proposed another stop criterion. For convenience, we call it Amplitude Ratio Stop Criteria.

Amplitude ratio stop criteria

(1) Find the upper envelope $U(t)$ and lower envelope $L(t)$.
(2) Introduce the mode amplitude $a(t) = [U(t) - L(t)]/2$.
(3) Calculate the evaluation function $\sigma(t) = |m(t)/a(t)|$.
(4) The sifting is stopped when $\sigma(t) < \theta_1$ for some prescribed fraction $(1 - \alpha)$ of the total duration and $\sigma(t) < \theta_2$ for the remaining fraction.

One typically set is $\alpha = 0.05$, $\theta_1 = 0.05$ and $\theta_2 = 10\theta_1$. This stop criteria compares the amplitude of the mean with the amplitude of the corresponding IMF. If the amplitude of the mean envelope is relatively small compared with the amplitude of the corresponding IMF at all data points, then stop sifting. We adopted this idea to the B-spline algorithm. Since the B-spline algorithm calculate the mean envelope without calculating the upper envelope $U(t)$ and lower envelope $L(t)$, instead of calculating the mode amplitude $a(t) = [U(t) - L(t)]/2$, we use B-Spline again and let

$$\begin{aligned} a(t) &= \sum \frac{1}{4}[|\, s(\tau_{j+1}) - s(\tau_{j+2})| + |s(\tau_{j+2}) - s(\tau_{j+3})\,|]B_{j,4,\tau}(t) \\ &= \sum \frac{1}{4}|\, s(\tau_{j+1}) - 2s(\tau_{j+2}) + s(\tau_{j+3})\,|B_{j,4,\tau}(t) \end{aligned}$$

where τ_j are the local extrema point of $s(t)$.

The experiments show that the mode amplitude function $a(t)$ defined by the B-Spline function works similarly to the difference between the upper envelope and lower envelope. The reason that we want to adopt the Amplitude ratio stop criteria is that it is aimed at guaranteeing globally small fluctuations in the mean while taking into account locally large excursions [Rilling *et al.* (2003)].

5. Numerical Results

In this section, we will compare the decomposition results of some examples by the modified B-spline method with ratio boundary extension proposed in Sec. 3 and Amplitude ratio stop criteria as in Sec. 4. We will compare the results with the original B-spline method [Riemenschneider *et al.* (2005); Chen *et al.* (2006)] the original Cubic Spline method [Rilling *et al.* (2003); Flandrin (2007)] and the modified Cubic Spline with ratio boundary extension.

Example 5.1: $s(t) = 10*\cos((\frac{t}{80})^{1.5}*\pi)+2*\cos((\frac{t}{100})^{0.8}*\pi)+(\frac{t}{500}+2),\ t = 20:2^{11}.$

Figure 5 is the graph of the original signal and its components. Ideally, this signal should be decomposed into two IMFs, $(10*\cos((\frac{t}{80})^{1.5}*\pi),\ 2*\cos((\frac{t}{100})^{0.8}*\pi))$ and a residue $(\frac{t}{500}+2)$. The two IMFs are Frequency modulated signals. Figure 6 shows the decomposition results of the four different methods. From the graph, we find

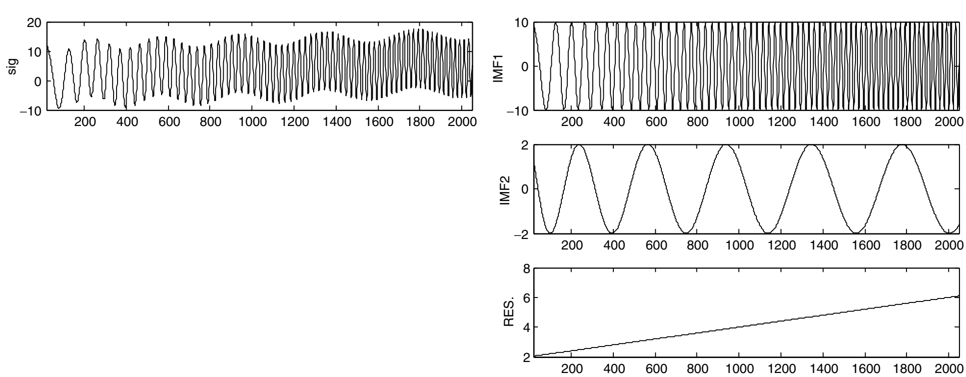


Fig. 5. Graph for Example 5.1: a signal with two FM components and a linear residue.

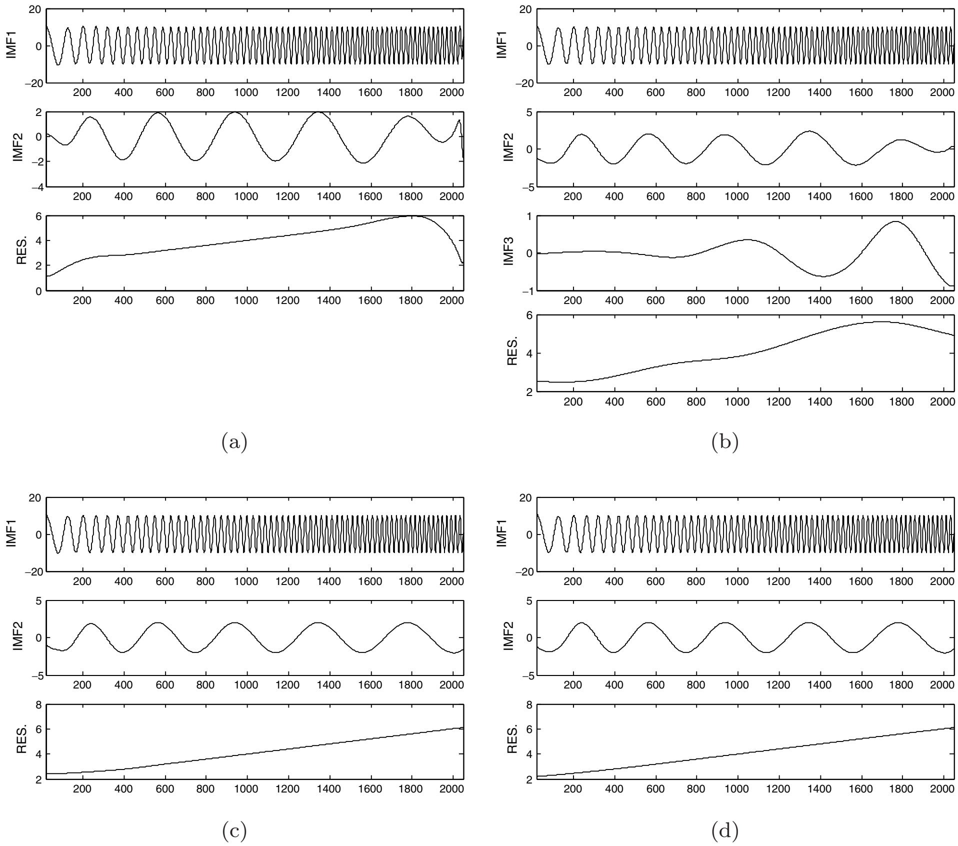


Fig. 6. (a) The IMFs of Example 5.1 by the original B-Spline method with symmetric boundary extension and original stop criteria. (b) The IMFs by Cubic Spline method with symmetric boundary extension and amplitude ratio stop criteria. (c) The IMFs by B-Spline method with ratio boundary extension and amplitude ratio stop criteria. (d) The IMFs by Cubic Spline with ratio boundary extension and amplitude ratio stop criteria.

that due to the boundary effect, the original B-spline method and the original Cubic Spline method cannot decompose the signal correctly from the second IMF, and we cannot get the correct residue. The modified Cubic Spline with the ratio boundary extension and the modified B-spline method with ratio boundary extension and amplitude ratio stop criteria decompose the signal almost perfectly.

Example 5.2: $s(t) = 5 * \cos((\frac{t}{100})^{1.8} * \pi) + (\frac{t}{200})^{1.5} * \cos(\frac{t}{180} * \pi) + (\frac{t}{200} - 5)^2,\ t = 20 : 2^{11}$.

Figure 7 is the graph of the original signal and its components. This signal has a frequency modulated component $5 * \cos((\frac{t}{120})^{1.8} * \pi)$, an amplitude modulated component $(\frac{t}{200})^{1.5} * \cos(\frac{t}{180} * \pi)$ and a quadratic residue $(\frac{t}{200} - 45)^2$. Again, if we compare the result of the four different algorithm as in Fig. 8, we find that the original B-spline method and the original Cubic Spline method over-decompose the

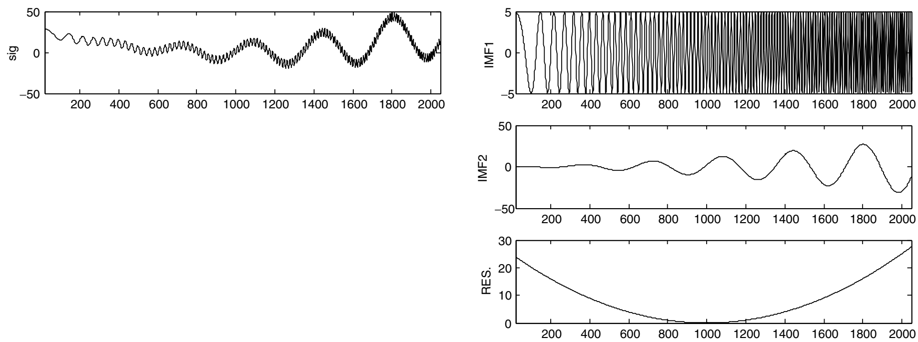


Fig. 7. Graph for Example 5.2: a signal with one FM component, one AM component and a linear residue.

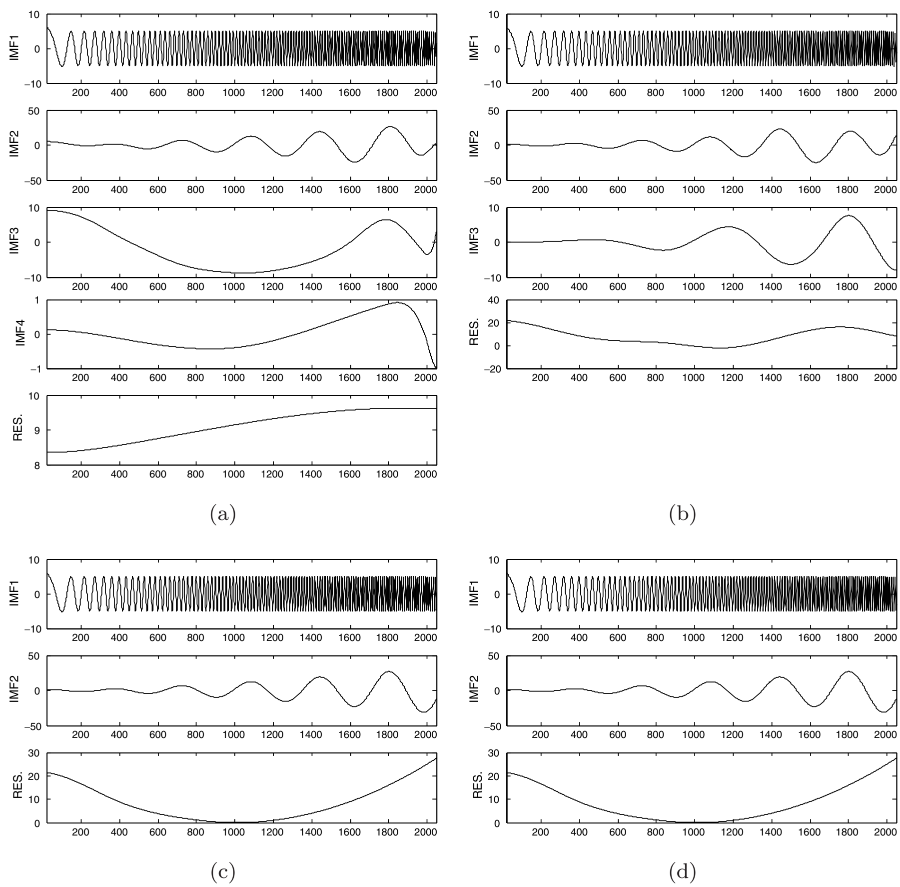


Fig. 8. (a) The IMFs of Example 5.2 by the original B-Spline method with symmetric boundary extension and original stop criteria. (b) The IMFs by Cubic Spline method with symmetric boundary extension and amplitude ratio stop criteria. (c) The IMFs by B-Spline method with ratio boundary extension and amplitude ratio stop criteria. (d) The IMFs by Cubic Spline with ratio boundary extension and amplitude ratio stop criteria.

signal, the right hand side of the second IMF is bad due to the improper boundary extension and the residue do not show any trend of the original signal. The modified B-spline method with ratio boundary extension and amplitude ratio stop criteria, and the modified Cubic Spline method with amplitude ratio stop criteria decompose the signal as two IMFs and one residue as expected.

Based on the numerical results in the above examples, we conclude that the ratio boundary extension and Amplitude ratio stop criteria indeed give us an improved implementation of the Empirical Mode Decomposition.

6. Conclusion

The Empirical Mode Decomposition (EMD) is a promising tool for the analysis of nonstationary and nonlinear signal processing. It has been applied with great success for nonlinear and nonstationary signal analysis in various areas.

This paper proposed some new ideas on the EMD algorithm. We proposed the ratio boundary extension which is more adaptive to the signal compared with the symmetric extension. We would like to emphasize the importance of the boundary extension. As mentioned in other papers, the influence of the ends will propagate into the data range in the low frequency components. And from the numerical experiments, we find that the proper data prediction is important for us to get the correct IMFs. If we make a wrong prediction at the first step, the whole signal will be decomposed incorrectly from the first step and it will ruin all the following EMD decomposition steps. So it is important to choose a proper boundary extension at every step. We also investigate the stop criteria and applied the Amplitude ratio stop criteria to our B-Spline algorithm.

Up to now, most of the progress with EMD is in its application. The decomposition is only defined as the output of an algorithm. The mathematical ground for EMD has not been set up yet. In the future, we will devote to the theoretical research on EMD.

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