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RECENT MATHEMATICAL DEVELOPMENTS ON EMPIRICAL MODE DECOMPOSITION

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Building the mathematical foundation for the empirical mode decomposition is an important issue in adaptive data analysis. The task of building such a foundation consists of two stages. The first is to construct a large bank of basis functions for the time–frequency analysis of nonlinear and nonstationary signals. The second is to establish a fast adaptive decomposition algorithm. We survey recent mathematical progress on these two stages. Related results on piecewise linear spectral sequences and the Bedrosian identity are also reviewed.

Keywords: Empirical mode decomposition; the Hilbert–Huang transform; intrinsic mode functions; mathematical foundation; spectral sequences; orthonormal bases; nonlinear phases; the Bedrosian identity.

1. Introduction

Since it was first introduced in Ref. 12, the empirical mode decomposition (EMD) has been found useful in many engineering areas. Recently, EMD related mathematical problems attracted much attention from the mathematical community.[7,16,22–26,29,31–34,36,37,39] These studies aim at better understanding the mathematical insight of the algorithm, building a reasonable mathematical foundation for the method and improving upon it. Establishing the mathematical foundation for EMD requires us to address two major issues. The first is the construction of a large bank of basis functions which are suitable for the time–frequency analysis of nonlinear and nonstationary signals. The second is the development of fast adaptive decomposition algorithms for the representation of a given signal by the basis functions. Although we have not obtained a complete answer to these issues, some interesting partial answers have become available as a result of the study which took place in the last few years. The purpose of this paper is to review the recent mathematical development in this interesting subject.

A fundamental problem in data analysis is to obtain an adaptive application-oriented representation for a given data set. EMD is an efficient method for such adaptive representations. Indeed, the original purpose of EMD is to decompose a signal into components, each of which has meaningful instantaneous frequency, and different components correspond to different frequency scales. EMD decomposes a signal into a finite sum of *intrinsic mode functions* (IMFs) based on the direct extraction of the energy associated with various intrinsic time scales. Many examples of using EMD show that the IMFs obtained from EMD provide physical insights which are crucial in engineering applications. Due to the fully adaptive nature of the method, it is particularly suitable for processing nonlinear and nonstationary signals.

We begin with a review of the notion of instantaneous amplitude and phase which are basic concepts in the time–frequency analysis of signals. If a signal f can be written as

$$f(t) = \rho(t)\cos\theta(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\rho \geq 0$, then we consider ρ and θ as the instantaneous amplitude and phase of f, respectively. However, in general, there exist many pairs of ρ and θ with $\rho \geq 0$ that satisfy decomposition (1.1).[21] A classical way of defining without ambiguity the instantaneous amplitude and phase of a real signal $f \in L^2(\mathbb{R})$ is through the Hilbert transform, which is defined for each function $f \in L^p(\mathbb{R}), 1 \leq p \leq \infty$, at $t \in \mathbb{R}$ as

$$(Hf)(t) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(s)}{t-s}ds := \frac{1}{\pi}\lim_{\substack{\varepsilon\to 0^+\\ N\to\infty}} \int_{\varepsilon\leq|t-s|\leq N} \frac{f(s)}{t-s}ds, \tag{1.2}$$

whenever the Cauchy principal value of the above singular integral exists. To form the *analytic signal*, we define

$$Af := f + iHf.$$

By the theory of the Hilbert transform,[5] Af has only non-negative Fourier frequencies. Then Af is further written as

$$(Af)(t) = \rho(t)e^{i\theta(t)}, \quad t \in \mathbb{R}.$$

Finally, the $\rho(t)$ and $\theta(t)$ above are defined as the instantaneous amplitude and phase of signal f at time t, respectively. The derivative θ' is regarded as the instantaneous frequency of f. This method for obtaining the instantaneous amplitude and frequency of signals is called the *analytic method*.

When the instantaneous frequency θ' is non-negative, we say that it is physically meaningful. Signals should have special properties to ensure that their instantaneous frequency obtained by the analytic method is physically meaningful. Therefore, we introduce a class of signals by setting

$$\mathcal{M} := \{f \in L^2(\mathbb{R})\colon f \text{ is real}, (Af)(t) = \rho(t)e^{i\theta(t)}, \rho \geq 0, \theta' \geq 0\}. \tag{1.3}$$

Recall that EMD aims at a decomposition of functions ψ having the properties that

(a) ψ has exactly one zero between any two consecutive local extrema;
(b) the local mean of ψ is zero.

Functions with properties (a) and (b) above are called intrinsic mode functions (IMFs) in Ref. 12. According to Ref. 12, empirically an IMF has physically meaningful instantaneous frequency. To provide more mathematical insight for IMFs, as a first step we consider functions in $\mathcal{M}$ as a basic atom for the Hilbert-Huang Transform (HHT). For consistence, functions in $\mathcal{M}$ are still referred to as intrinsic mode functions.

Building a mathematical foundation for EMD consists of two steps. The first is to formulate mathematical characterizations of the functions in $\mathcal{M}$ and construct a large bank of functions in $\mathcal{M}$ with explicit expressions. The second is to develop an adaptive and fast algorithm $\mathcal{A}$ for decomposition of a real function $f \in L^2(\mathbb{R})$ into a monotone function and a sum of functions in the bank constructed in the first step with the summand decaying fast. It seems that EMD is a numerical approximation of the ideal algorithm $\mathcal{A}$.

By far there are few mathematical efforts on the second step. A better understanding of EMD from the study of its variations and extensions may provide some insight into the ideal algorithm $\mathcal{A}$. In Sec. 2, we introduce the recently developed one-dimensional B-spline EMD[7] and two-dimensional finite element based EMD.[36]

For the first step, the problem for constructing elements in $\mathcal{M}$ was proposed by the first author in 2002. We are interested in finding $\rho \in L^2(\mathbb{R})$ and $\theta \in C^1(\mathbb{R})$ that satisfy the nonlinear singular integral equation

$$[H(\rho(\cdot)\cos\theta(\cdot))](t) = \rho(t)\sin\theta(t), \quad t \in \mathbb{R} \tag{1.4}$$

subjected to the constraint

$$\rho(t) \geq 0, \ \frac{d\theta(t)}{dt} \geq 0, \quad t \in \mathbb{R}.$$

Along this line, the recent work[22] provides some mathematical insight to this problem. In Sec. 3, we focus on the construction of functions in $\mathcal{M}$. A mathematical characterization[29] of property (a) will be discussed. Our main interest is in recent results on the singular integral equation (1.4) from Refs. 22, 24, 31 and 39.

In some sense, intrinsic mode functions can be viewed as basis functions for signal analysis. They generally have nonconstant frequencies. This property partly accounts for the importance and efficiency of IMFs in the time–frequency analysis of nonlinear and nonstationary signals. It also suggests the study of orthonormal bases for the function space $L^2(A)$ with nonconstant frequencies, where A is a Lebesgue measurable subset of $\mathbb{R}$. We shall review in Sec. 4 the results of Ref. 16 on orthonormal bases

$$f_n := e^{2\pi i g_n}, \quad n \in \mathbb{Z}, \tag{1.5}$$

for $L^2([0,1])$, where the phase functions g_n are piecewise linear on $[0,1]$, and those in Refs. 8, 24 and 31 on orthonormal bases with smooth nonlinear phases for $L^2(\mathbb{R})$ and $L^2([0,2\pi])$.

The Bedrosian identity is an important mathematical formula in signal analysis. In particular, it is a useful tool for HHT. A study of the identity helps us better understand the condition for the following important equality[9,12,20,21]

$$[H(\rho(\cdot)e^{i\theta(\cdot)})](t) = \rho(t)H(e^{i\theta(\cdot)})(t).$$

Moreover, it contributes to the construction of functions in $\mathcal{M}$. To see this, we suppose that a unimodular real signal $\cos\theta(\cdot)$ satisfies

$$[H(\cos\theta(\cdot))](t) = \sin\theta(t), \quad t \in \mathbb{R}.$$

By finding a non-negative function $\rho \in L^2(\mathbb{R})$ such that

$$H(\rho\cos\theta(\cdot)) = \rho H(\cos\theta(\cdot)),$$

we obtain a new function $\rho\cos\theta(\cdot) \in \mathcal{M}$. In Sec. 5, we review recent results in Refs. 25, 31–33 and 37–39 on the Bedrosian identity. Especially, we present necessary and sufficient conditions for the Bedrosian identity to hold. We draw conclusive remarks in Sec. 6.

2. The B-Spline EMD and 2-D Finite Element EMD

We review in this section the one-dimensional B-spline EMD and two-dimensional finite element based EMD developed in Refs. 7 and 36, respectively. In the EMD algorithm, an important issue is the construction of the local mean function of a given signal. The original EMD uses cubic spline interpolation to construct the upper envelop function and lower envelop function of the signal. They are then used to construct the local mean. However, overshooting of the envelops is often observed in practical computation. This is because interpolation is not a good method to compute the envelop functions. To be more specific, we define envelop functions. For a given class A of functions defined on interval $[a,b]$ and a given function f not in the class A the upper envelop of f with respect to the class A is defined by

$$u(x) := \inf\{g(x) \colon g(y) \geq f(y), y \in [a,b], \text{ for all } g \in A\}, \quad x \in [a,b].$$

Likewise, the lower envelop of f with respect to the class A is defined by

$$\ell(x) := \sup\{g(x) \colon g(y) \leq f(y), y \in [a,b], \text{ for all } g \in A\}, \quad x \in [a,b].$$

Due to these definitions, it is not surprising that the cubic spline interpolation sometimes provide unsatisfactory results for the envelops since it is not realistic to expect interpolation gives good approximation for minimization or maximization. To overcome this difficulty, an alternative method for generating a local mean function was suggested in Refs. 7 and 36. The main purpose of this section is to discuss this alternative approach.

2.1. *A general setting for EMD*

We begin with the general setting for EMD described in Ref. 36. This generalization helps us understand the intrinsic properties of the EMD method and enables us to present the B-spline EMD and 2-D finite element based EMD in a unified way. Let $f \in L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$. We require to have a compactly supported basis ϕ_j, $j \in \mathbb{Z}$, for a subspace of $L^2(\mathbb{R}^d)$. The subspace is adapted to the given data f. As a result, the basis functions are also adapted to the given data f and will be used to construct the local mean surface of f. We also need the set $\mathcal{P} := \{p_j : j \in \mathbb{Z}\}$ of points in a domain in $\mathbb{R}^d$, which we call the *characteristic* points of f. They capture certain features of the given data. For example, they include the local extreme points for the 1-D EMD, and the local extreme and saddle points for the 2-D EMD. Associated with the set of characteristic points we define a smoothed linear functional of $f(p_j)$ by

$$\lambda(p_j) := S *' f(p_j), \tag{2.1}$$

where S is a generalized low-pass filter and $*'$ represents a generalized filtering operation. Mathematically, $\lambda(p_j)$ is a linear functional using values of f at the characteristic points near p_j. Basically, we specify the generalized low-pass filter as a set of positive-valued weights and the generalized filtering as a weighted sum of $f(p_j)$ and the function values of f at the neighboring characteristic points near p_j. In this way, the produced functional $\lambda(p_j)$ will be smoother than $f(p_j)$ since the local variation of $f(p_j)$ is averaged.

The essence of the EMD method is to subtract the local mean from the data so as to decompose the data into a high frequency and a low frequency component, namely, the local mean. Using the adaptive basis functions ϕ_j, $j \in \mathbb{Z}$, and the local smoothed functionals, we define the *local mean* of the data f as

$$m(p) := \sum_{j \in \mathbb{Z}} \lambda(p_j)\phi_j(p). \tag{2.2}$$

Since ϕ_j has a compact support and the functional $\lambda(p_j)$ is defined locally, the local mean capture the local feature of the given data. Note that we avoid using the "upper envelop" and "lower envelop" in the definition of the local mean because mathematically they are not well-defined by using spline interpolation, as we explained earlier.

The EMD method decomposes a signal into a finite sum of IMFs. In a general case, in particular in the 2-D case, we will not impose a specific definition of an

IMF. It should be determined by a specific stopping condition in the sifting process for a specific application. We now describe a general EMD algorithm. We extract the first intrinsic mode function by the following steps:

(1) Find the characteristic points p_j of f and compute values $f(p_j)$.
(2) Compute the smoothed set $\lambda(p_j)$ using Eq. (2.1).
(3) Compute the local mean m using Eq. (2.2).
(4) Compute $h = f - m$. If h satisfies a given stopping condition, stop. Otherwise, treat h as the data and iterate on h.

The output of this algorithm is the first IMF and we denote it by c_1 and specify $r_1 := f - c_1$ as the first residue. By applying the above procedure to the first residue r_1 we obtain the second IMF c_2. Repeat this process until a satisfactory result is obtained. The procedure generates a sequence of IMFs $c_1, c_2, \ldots, c_N$ and a residue function r_N if it converges. All numerical results confirm that the algorithm converges though it has not been proved mathematically. The basic idea of the EMD is to decompose a signal into the sum of IMFs with different scales and the residue function so that c_1 catches the highest frequency of f, c_2 the second highest frequency of f and r_N the lowest frequency of f. Specifically, this procedure yields a decomposition

$$f = \sum_{j \in \mathbb{N}_N} c_j + r_N, \tag{2.3}$$

where $\mathbb{N}_n := \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. We shall also use $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ and $\mathbb{Z}_+ := \{j \in \mathbb{Z} : j \geq 0\}$.

A different way of defining the local mean gives a different method for construction of EMD. The local mean can be defined by interpolation, by quasi-interpolation, or by other approximation approaches. In fact this general setting covers the envelop approach, the original EMD presented in Ref. 12, the B-spline approach in Ref. 7 and the 2-D EMD by using finite elements in Ref. 36. Specific description for the B-spline EMD and the finite element EMD will be given later. Here we only comment on the relation of the general setting and the original envelop EMD. For the original EMD developed in Ref. 12, the basis functions are cubic splines determined by the extreme points of the signal and the smoothed functional is the coefficients of the cubic splines defined by the cubic spline interpolation. In this case, both basis functions and the smoothed set are determined implicitly. Specifically, following Ref. 12, the local mean of the given data f is defined by the average of the upper envelop and the lower envelop. Recalling that both envelops are expressed by the cubic spline interpolation, they may be written as a linear combination of the B-spline basis. Their coefficients in fact are certain smoothed functionals of f at the local minima and maxima. Thus, the local mean so constructed falls into the general setting that we just described. The general setting allows us to view the EMD in a more general point of view.

2.2. The one-dimensional B-spline EMD

In the B-spline EMD, the basis functions and the smoothed functionals are defined explicitly, both being adapted to the given data f. For a given signal f, let $\sigma^f := \{\sigma_j^f : j \in \mathbb{Z}\}$ be its extreme points. Note that in this case we choose the characteristic points as the extreme points of f. The basis functions are chosen accordingly as B-splines B_{j,k,σ^f} defined by the kth order divided difference

$$B_{j,k,\sigma^f}(t) := (\sigma_{j+k}^f - \sigma_j^f)[\sigma_j^f, \ldots, \sigma_{j+k}^f](\cdot - t)_+^{k-1}, \quad t \in \mathbb{R},$$

where $[\sigma_j, \ldots, \sigma_{j+k}]g$ denotes the kth order divided difference of function g at nodes $\sigma_j, \ldots, \sigma_{j+k}$. B-splines can also be generated recursively.[4] Moreover, it is proved in Ref. 7 that the Hilbert transform of B-splines has the same recurrence as the B-splines. Along this line, the translation invariant operators which preserve the B-spline recurrence are characterized in Ref. 18.

The generalized low-pass filter is defined as the binomial sequence and the smoothed sequence of the local extrema is the moving average

$$\lambda(\sigma_j^f) := \frac{1}{2^{k-2}} \sum_{l \in \mathbb{N}_{k-1}} \binom{k-1}{l} f(\sigma_{j+l}^f).$$

The local mean of f is hence given by

$$V_{\sigma^f,k} f := \sum_{j \in \mathbb{Z}} \lambda(\sigma_j^f) B_{j,k,\sigma^f}.$$

The local mean defined in this way has certain advantages over the approaches using envelopes. It overcomes the overshooting problem of the upper and lower envelopes.[7] Moreover, it does not need to solve linear systems which the original EMD must do.

By the general EMD algorithm, the first IMF of a given signal f is obtained as follows: let $f_{1,0} := f$ and compute for $j = 1, 2, \ldots,$

$$f_{1,j} := f_{1,j-1} - V_{\sigma^{f_{1,j-1}},k} f_{1,j-1}$$

until

$$\sum_t \frac{|f_{1,j-1}(t) - f_{1,j}(t)|}{f_{1,j-1}^2(t)} < SD,$$

where SD is typically set between 0.2 and 0.3. The last term $f_{1,j}$ is the desired first IMF of f.

The B-spline EMD is an alternative method for generation of IMFs which avoids using envelops. Computationally, it does not have to construct two interpolations (upper envelop and lower envelop) which require to solve linear systems. It only use multiplications of the basis functions with the functionals. Hence, it requires significantly less computational cost than the original EMD. It has been demonstrated in Refs. 7, 15 and 26 by simulated examples and engineering applications that the B-spline EMD has a comparable performance with the original EMD.

2.3. The two-dimensional finite element based EMD

We now turn to a discussion of the two-dimensional EMD. In this case, the basis functions are smoothed linear finite element shape functions and the generalized low-pass filter is chosen as a weighted average of the function values at the characteristic points around the point of interest, where the characteristic points are chosen as the local extreme and saddle points of the given data f. This construction is a natural extension of the 1-D B-spline EMD to the 2-D case. It also overcomes the same overshooting problem arising in using the upper and lower envelopes of interpolation for the construction of the local mean surface. Another reason to avoid using 2-D interpolation is that its computational cost can be huge.

We now review the 2-D finite element based EMD developed in Ref. 36. Suppose that $\Omega \subseteq \mathbb{R}^2$ is a closed polygonal domain and $f \in L^2(\Omega)$. Let $\mathbb{I}$ denote an appropriate index set and we denote by $\Delta := \{p_j \in \Omega\colon j \in \mathbb{I}\}$ the collection of the characteristic points of f. By using the Delaunay method,[28] the domain Ω is partitioned into a triangular mesh with vertices being the characteristic points. In this triangular mesh, any triangle does not overlap with any other triangles in the mesh, and a vertex of a triangle is not in the interior of an edge of another triangle in the mesh. Hence, if $p_j \in \Delta$ is not on the boundary of Ω, then there are a finite number of points $\Delta_j \subseteq \Delta$ such that they are the vertices of the polygon surrounding p_j and no other points of Δ except p_j located interior to the polygon. For $p_j \in \Delta$, we let T_ℓ denote a triangle with the vertex p_j and two other points in Δ_j and let k_j be the cardinality of Δ_j. The polygon P_j around p_j has the form

$$P_j = \bigcup_{\ell \in \mathbb{N}_{k_j}} T_\ell.$$

When $p_j \in \Delta$ is on the boundary of Ω, we need to extend the domain Ω appropriately so that p_j is an interior point of the extended domain.

The finite element basis functions are constructed associated with the triangle partition. Mainly, to each $j \in \mathbb{I}$, we assign a basis shape function ϕ_j in the following manner: outside the polygon P_j, ϕ_j is equal to zero and on each T_ℓ it is a linear polynomial satisfying the interpolation conditions

$$\phi_j(p) = \begin{cases} 1, & \text{if } p = p_j, \\ 0, & \text{if } p \in \Delta_j. \end{cases} \tag{2.4}$$

Clearly, the function $\phi_j \in C(\Omega)$ is a piecewise linear polynomial defined on Ω supported on the polygon P_j. They are finite element shape functions and have been used extensively in numerical solutions of partial differential equations and computer aided geometric design. Following the general setting we need smoothed functionals $\lambda(p_j)$, which are chosen in this case as a weighted average of the values of the signal f at the neighboring characteristic points of the point p_j. A specific example of the smoothed functional is as follows:

$$\lambda(p_j) := \alpha f(p_j) + (1-\alpha)\frac{1}{k_j}\sum_{p\in\Delta_j} f(p), \tag{2.5}$$

where the parameter α controls the degree of the smoothing and it is chosen empirically.

A local mean surface is constructed accordingly by setting

$$\tilde{m} := \sum_{p_j\in\Delta} \lambda(p_j)\phi_j. \tag{2.6}$$

To see the locality of the mean function defined above, we let T be a triangle element with vertices p_i, p_j and p_k of the triangularization for domain Ω. It is clear that due to the compact support of the shape functions, only three shape functions ϕ_i, ϕ_j and ϕ_k are nonzero at an interior point of T. In particular, for $p \in T$, we observe that

$$\tilde{m}(p) = \lambda(p_i)\phi_i(p) + \lambda(p_j)\phi_j(p) + \lambda(p_k)\phi_k(p). \tag{2.7}$$

This locality leads to a fast algorithm for computation, since the mean $\tilde{m}$ can be evaluated locally. Note that however, the surface generated by Eq. (2.6) or (2.7) is continuous but not smooth (i.e. it is in C^0 but not in C^1). This may consequently introduce additional sharp structures into the data. To avoid additional new oscillations that may introduce artificial frequency information, a smooth local mean m is desirable for application purposes. We may choose to use smooth higher order finite elements to overcome this problem. Using higher order finite elements may result in significant increase of the computational cost. An alternative idea was proposed in Ref. 36 which is to apply a smoothing filter to $\tilde{m}$ in order to obtain a new smooth mean m, avoiding the use of higher order finite elements. Namely, we will use a specially designed bi-cubic spline interpolation m of $\tilde{m}$ so that the new mean m is smooth, it preserves the crucial properties of $\tilde{m}$, and the additional computational cost is as small as possible. The interested readers are referred to Ref. 36 for the construction of the smoothing filter.

The 2-D finite element based EMD has been confirmed by numerical studies to be a useful and efficient algorithm. Numerical experiments using both simulated and practical texture images in Ref. 36 show that it is able to separate components of different scales from images. Moreover, it was used in Ref. 36 to detect defects in raw textiles. More recent interesting developments in multi-dimensional EMD are found in Refs. 3, 14, 19 and 35.

3. Characterizations and Constructions of IMFs

Many research results[12] show that the Fourier transform is not suitable for the time–frequency analysis of nonlinear and nonstationary signals. In practice, the Fourier transform requires that the signal under consideration be stationary and linear.[12]

It was observed in Ref. 12 that comparing with nonstationarity, nonlinearity of signals affects much more the soundness of the time–frequency analysis by the Fourier transform. A promising method for the time–frequency analysis of stationary but nonlinear signals is through the *circular Hilbert transform*, which is defined for each $g \in L^1_{2\pi}$ at $t \in [0, 2\pi]$ as

$$(\tilde{H}f)(t) := \text{p.v.}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\cot\frac{s}{2}ds := \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\varepsilon\le|s|\le\pi} f(t-s)\cot\frac{s}{2}ds$$

if the Cauchy principal value of the above singular integral exists. Here $L^p_{2\pi}$, $1 \le p \le \infty$, denotes the set of the 2π-periodic functions g whose restriction in $[0, 2\pi]$ belongs to $L^p([0, 2\pi])$. To give a brief presentation of the method, we introduce

$$\widetilde{\mathcal{M}} := \{f \in L^2_{2\pi}\colon f \text{ is real}, (\tilde{A}f)(t) = \rho(t)e^{i\theta(t)},\ \rho \ge 0,\ \theta' \ge 0\}, \tag{3.1}$$

where for each real function $f \in L^2_{2\pi}$,

$$\tilde{A}f := f + i\tilde{H}f.$$

If a real signal $f \in L^2_{2\pi}$ can be decomposed into a finite sum of functions in $\widetilde{\mathcal{M}}$ then a time–frequency–energy distribution of f can be formed by applying the circular Hilbert transform to each decomposed function in $\widetilde{\mathcal{M}}$.

It is worthwhile to point out that properties (a) and (b) stated in Sec. 1 are not sufficient for a 2π-periodic function to belong to $\widetilde{\mathcal{M}}$. This fact was discovered in Ref. 29, where it was proved that a real function $f \in C^2[a, b]$ has property (a) if and only if it is a solution of a self-adjoint ordinary differential equation

$$\frac{d}{dt}\left(P\frac{df}{dt}\right)(t) + Q(t)f(t) = 0, \quad t \in (a, b),$$

where $P \in C^1[a, b]$, $Q \in C[a, b]$ are strictly positive. Several examples of 2π-periodic functions were constructed in Ref. 29. Those functions are of properties (a) and (b) but not in $\widetilde{\mathcal{M}}$.

Similar to that for $\mathcal{M}$, a way of construction of functions in $\widetilde{\mathcal{M}}$ is to solve the nonlinear singular integral equation

$$[\tilde{H}(\rho(\cdot)\cos\theta(\cdot))](t) = \rho(t)\sin\theta(t), \quad t \in [0, 2\pi] \tag{3.2}$$

subjected to

$$\rho(t) \ge 0, \quad \frac{d\theta(t)}{dt} \ge 0, \quad t \in [0, 2\pi]. \tag{3.3}$$

The main purpose of this section is to introduce recent results on singular integral equations (1.4) and (3.2) from Refs. 22, 25, 31, 34 and 39.

We need some preliminaries on Hardy spaces.[10,11,27] We shall use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. Let $\mathbb{C}_+ := \{z \in \mathbb{C}\colon \Im(z) > 0\}$, $D := \{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$. Denote by $\mathbf{H}(D)$ and $\mathbf{H}(\mathbb{C}_+)$ the set of all the holomorphic functions on D and $\mathbb{C}_+$, respectively. We introduce the Hardy spaces by setting for $0 < p < \infty$

$$\mathbf{H}^p(D) := \left\{ f \in \mathbf{H}(D)\colon \sup\left\{ \int_0^{2\pi} |f(re^{it})|^p dt\colon r \in (0,1) \right\} < \infty \right\}$$

and

$$\mathbf{H}^p(\mathbb{C}_+) := \left\{ f \in \mathbf{H}(\mathbb{C}_+)\colon \sup\left\{ \int_{\mathbb{R}} |f(x+iy)|^p dx\colon y > 0 \right\} < \infty \right\}.$$

For $p = \infty$, we let

$$\mathbf{H}^\infty(D) := \{ f \in \mathbf{H}(D)\colon \sup\{|f(z)|\colon z \in D\} < \infty \}$$

and

$$\mathbf{H}^\infty(\mathbb{C}_+) := \{ f \in \mathbf{H}(\mathbb{C}_+) : \sup\{|f(z)| : z \in \mathbb{C}_+\} < \infty \}.$$

Each function $f \in \mathbf{H}^p(D)$ or $\mathbf{H}^p(\mathbb{C}_+)$, $0 < p \le \infty$, has a nontangential boundary limit in $L^p(\mathbb{T})$ or $L^p(\mathbb{R})$, respectively. The boundary limit is still denoted by f. With this convention, we call a function $f \in \mathbf{H}^\infty(D)$ an inner function provided that $|f| = 1$ almost everywhere on $\mathbb{T}$. An interesting class of inner functions on D is the Blaschke products. Such functions are given by

$$B(z) := z^k \prod_{n \in \mathbb{N}} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \quad z \in D,$$

where $k \in \mathbb{Z}_+$, $\{z_n : n \in \mathbb{N}\} \subseteq D \backslash \{0\}$ satisfies that

$$\sum_{n \in \mathbb{N}} (1 - |z_n|) < \infty.$$

A characterization for inner functions on D can be found in Refs. 10, 11 and 27. The Blaschke product and inner function on $\mathbb{C}_+$ are obtained from their counterparts on D through the Cayley transform

$$\mathcal{K}(w) := \frac{i - w}{i + w}, \quad w \in \mathbb{C}_+,$$

which is a conformal mapping from $\mathbb{C}_+$ to D.

The subsequent characterizations of the singular integral equations (1.4) and (3.2) were given in Ref. 22.

Theorem 3.1. *Let* $1 \le p \le \infty$, $\rho \in L^p_{2\pi}$ *be real and* θ *a real Lebesgue measurable function on* $[0, 2\pi]$. *Then* ρ, θ *satisfy the singular integral equation* (3.2) *if and only if*

$$\rho(t) e^{i\theta(t)} = f(e^{it}), \quad t \in [0, 2\pi]$$

for some $f \in \mathbf{H}^p(D)$ *with* $\Im f(0) = 0$. *In particular, for the unimodular case that* $\rho \equiv 1$, (3.2) *holds if and only if* $e^{i\theta(\cdot)}$ *is the boundary value of an inner function* f *on* D *such that* $\Im f(0) = 0$.

Theorem 3.2. *Let $1 \le p < \infty$, $\rho \in L^p(\mathbb{R})$ and θ be a Lebesgue measurable function on $\mathbb{R}$. Then ρ and θ satisfy singular integral equation* (1.4) *if and only if $\rho e^{i\theta(\cdot)}$ is the boundary value of some function in $\mathbf{H}^p(\mathbb{C}_+)$.*

A definition of the Hilbert transform of functions in $L^\infty(\mathbb{R})$ using harmonic representations of distributions was proposed in Ref. 22. Under this definition, similar results as those for Eq. (3.2) hold for Eq. (1.4) when $p = \infty$. More details for this case can be found in Ref. 22.

Solutions of (1.4) and (3.2) with explicit expression are desirable in engineering applications. In the unimodular case, an important class of phases θ with explicit form satisfying Eq. (1.4) or (3.2) are provided by finite Blaschke products (see, for example, Ref. 27). Here, we consider those on $[0, 2\pi]$ of the form

$$e^{i\theta(t)} = \prod_{j \in \mathbb{N}_n} \frac{e^{it} - \lambda_j}{1 - \lambda_j e^{it}}, \quad t \in [0, 2\pi], \tag{3.4}$$

where $n \in \mathbb{N}$, $\lambda_j \in [0, 1)$, $j \in \mathbb{N}_n$, and phase functions θ on $\mathbb{R}$ determined by

$$e^{i\theta(t)} = \frac{1 + it}{\sqrt{1 + t^2}} \prod_{j \in \mathbb{N}_n} \frac{e^{i2\arctan t} - \lambda_j}{1 - \lambda_j e^{i2\arctan t}}, \quad t \in \mathbb{R}. \tag{3.5}$$

It can be verified that θ given above have a non-negative first derivative. This implies that the instantaneous frequency of $\cos\theta(\cdot)$ obtained by the Hilbert transforms is physically meaningful.

Let θ be specified explicitly by Eq. (3.4) or (3.5). Functions $\rho \in L^2_{2\pi}$ or $L^2(\mathbb{R})$ satisfying Eq. (3.2) or (1.4) can be obtained by solving the Bedrosian identity

$$\tilde{H}(\rho\cos\theta(\cdot)) = \rho\tilde{H}(\cos\theta(\cdot)) \tag{3.6}$$

or

$$H(\rho\cos\theta(\cdot)) = \rho H(\cos\theta(\cdot)). \tag{3.7}$$

This was carried out in Ref. 24 based on new necessary and sufficient characterizations for the Bedrosian identities. We summarize the obtained results below.

Proposition 3.3. *Let θ be defined by* (3.4) *with $\lambda_j \in [0, 1)$, $j \in \mathbb{N}_n$. Then a real $\rho \in L^2_{2\pi}$ satisfies* (3.6) *if and only if there exists $b_j \in \mathbb{C}$, $j \in \mathbb{Z}_{n-1}$ and $b_{n-1}, c \in \mathbb{R}$, such that*

$$\rho(t) = \Re\left(\frac{e^{it}\sum_{j \in \mathbb{Z}_n} b_j e^{ijt}}{\prod_{j \in \mathbb{N}_n}(1 - \lambda_j e^{it})}\right) + c, \quad t \in [0, 2\pi].$$

Proposition 3.4. *Let θ be given by* (3.5) *with $\lambda_j \in [0, 1)$, $j \in \mathbb{N}_n$. Then a real function $\rho \in L^2(\mathbb{R})$ satisfies* (3.7) *if and only if there exists $b_j \in \mathbb{C}$, $j \in \mathbb{N}_n$ and $c \in \mathbb{R}$ such that*

$$\rho(t) = \frac{1}{\sqrt{1 + t^2}}\left(\Re\left(\frac{\sum_{j \in \mathbb{N}_n} b_j e^{i2j\arctan t}}{\prod_{j \in \mathbb{N}_n}(1 - \lambda_j e^{i2\arctan t})}\right) + c\right), \quad t \in \mathbb{R}.$$

Similar results to those in the above two propositions were obtained in Ref. 31. Solutions ρ of Eqs. (3.6) and (3.7) when the phase function θ is defined by a single Blaschke product were first discovered in Ref. 39.

4. Orthonormal Bases with Nonconstant Frequencies

Motivated by Huang's work[12] on representing nonlinear and nonstationary signals by adaptive decomposition, the recent papers[8,16,24,31] aim at developing orthonormal bases for $L^2(A)$ with nonconstant frequencies, where A is a Lebesgue measurable subset of $\mathbb{R}$. We start with reviewing the orthonormal bases (1.5) for $L^2([0,1])$ constructed in Ref. 16 that have piecewise constant frequencies.

The basic concept introduced in Ref. 16 is the spectral sequence. A sequence of real-valued functions g_n, $n \in \mathbb{Z}$, defined on $[0,1]$, is called a spectral sequence of $[0,1]$ if the exponential function system f_n, $n \in \mathbb{Z}$, defined by (1.5) in terms of g_n is an orthonormal basis for $L^2([0,1])$. In some sense, the EMD is an adaptive numerical method for the construction of spectral sequences.

The piecewise linear spectral sequence g_n with the knot at $1/2$ was constructed in Ref. 16. Specially, it was characterized in Ref. 16 the condition for the phase functions having the form

$$g_n(t) := \begin{cases} a_n t + b_n, & t \in [0, \frac{1}{2}), \\ c_n t + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \tag{4.1}$$

where $a_n, b_n, c_n, d_n \in \mathbb{R}$, to be a spectral sequence of $[0,1]$. We present below a main result in Ref. 16.

Theorem 4.1. *Suppose that g_n, $n \in \mathbb{Z}$, is defined by Eq.* (4.1) *with $g_0 = 0$ and let $G := \{g_n : n \in \mathbb{Z}\}$. If the cardinality $\#\{n \in \mathbb{Z}\backslash\{0\} : a_n = 0\} > 0$ then g_n, $n \in \mathbb{Z}$, is a spectral sequence of $[0,1]$ if and only if $G = \{u_n, v_n : n \in \mathbb{Z}\}$, where u_n, v_n are defined by*

$$u_n(t) := \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ c_n t + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \quad v_n(t) := \begin{cases} 2nt + b'_n, & t \in [0, \frac{1}{2}), \\ c_n t + d'_n, & t \in [\frac{1}{2}, 1], \end{cases}$$

with constants b_n, c_n, d_n, b'_n and d'_n satisfying the conditions that $\{c_n : n \in \mathbb{Z}\} = 2\mathbb{Z}$, $c_n \neq c_m$ for $n \neq m$ and $(b'_n - d'_n) - (b_n - d_n) \in \mathbb{Z} + \frac{1}{2}$. If $\#\{n \in \mathbb{Z}\backslash\{0\} : a_n = 0\} = 0$ and $a_n = c_n$, $n \in \mathbb{Z}$, then g_n, $n \in \mathbb{Z}$, is a spectral sequence of $[0,1]$ if and only if $G = \{u_n, v_n : n \in \mathbb{Z}\}$, where u_n and v_n are defined by

$$u_n(t) := \begin{cases} 2nt + b_n, & t \in [0, \frac{1}{2}), \\ 2nt + d_n, & t \in [\frac{1}{2}, 1], \end{cases} \quad v_n(t) := \begin{cases} (2n + c)t + b'_n, & t \in [0, \frac{1}{2}), \\ (2n + c)t + d'_n, & t \in [\frac{1}{2}, 1], \end{cases}$$

with constants b_n, d_n, b'_n, d'_n and c satisfying the conditions that $b_n - d_n \in \mathbb{Z}$, $b'_n - d'_n \in \mathbb{Z} + (1 - c)/2$ and $c \in \mathbb{R}\backslash 2\mathbb{Z}$.

It was also discovered in Ref. 16 that the spectral sequences g_n, $n \in \mathbb{Z}$, of form

$$g_n(t) := \begin{cases} a_n t + b_n, & t \in [0, \theta), \\ c_n t + d_n, & t \in [\theta, 1], \end{cases} \tag{4.2}$$

where $a_n, b_n, c_n, d_n \in \mathbb{R}$ and $\theta \in (0, 1)$, cannot be continuous except for the classical case.

Theorem 4.2. *Suppose that* g_n, $n \in \mathbb{Z}$, *defined by* (4.2) *with* $g_0 = 0$ *is a continuous spectral sequence of* $[0, 1]$. *Let* $G := \{g_n : n \in \mathbb{Z}\}$. *Then* $G = \{h_n : n \in \mathbb{Z}\}$ *where* $h_n(t) = nt + b_n$, $t \in [0, 1]$, $b_n \in \mathbb{R}$, $n \in \mathbb{Z}$.

The classical Walsh system can be constructed from a special piecewise constant spectral sequence of $[0, 1]$. This was done in Ref. 16 by setting $g_0 = 0$ on $[0, 1]$ and for $j \in \mathbb{Z}_{2^n}$, $n \in \mathbb{N}$ recursively

$$g_{2^n+j}(t) := \begin{cases} g_j(t), & t \in [t_{n,2k}, t_{n,2k+1}), \quad k \in \mathbb{Z}_{2^n}, \\ g_j(t) + \frac{1}{2}, & t \in [t_{n,2k+1}, t_{n,2k+2}), \quad k \in \mathbb{Z}_{2^n}, \end{cases}$$

where

$$t_{n,k} := \frac{k}{2^{n+1}}, \quad k \in \mathbb{Z}_{2^{n+1}+1}.$$

Theorem 4.3. *Let* g_n, $n \in \mathbb{Z}_+$, *be given as above. Then* g_n, $n \in \mathbb{Z}_+$, *is a spectral sequence of* $[0, 1]$ *and* f_n, $n \in \mathbb{Z}_+$, *given by*

$$f_n(t) := e^{2\pi i g_n(t)}, \quad t \in [0, 2\pi],$$

is the Walsh system on $[0, 1]$.

The spectral sequences constructed in Ref. 16 are piecewise linear and discontinuous. Other classes of nonlinear spectral sequences are desirable. We are particularly interested in constructing smooth nonlinear spectral sequences, hoping that they are better than linear spectral sequence of the classical Fourier basis in representing a nonlinear signal.

Set $L_r^2(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : f \text{ is real}\}$. For the purpose of decomposing an arbitrary function in $L_r^2(\mathbb{R})$ into a sum of functions in $\mathcal{M}$, Refs. 24 and 31 studied the construction of orthonormal bases for $L_r^2(\mathbb{R})$ with the basis functions coming from $\mathcal{M}$. To this end, it was first observed in Ref. 24 that such constructions can be reformulated into the constructions of orthonormal bases for $\mathbf{H}^2(\mathbb{C}_+)$. Here we note that $\mathbf{H}^2(\mathbb{C}_+)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{\mathbf{H}^2(\mathbb{C}_+)} := \int_{\mathbb{R}} f(t)\overline{g(t)}dt, \quad f, g \in \mathbf{H}^2(\mathbb{C}_+).$$

Theorem 4.4. *Let $\rho_j \in L^2(\mathbb{R})$ be non-negative and θ_j real Lebesgue measurable functions on $\mathbb{R}_+$, $j \in \mathbb{Z}_+$. Functions $\rho_j \cos\theta_j$, $\rho_j \sin\theta_j$, $j \in \mathbb{Z}_+$, satisfy*

$$H(\rho_j(\cdot)\cos\theta_j(\cdot))(t) = \rho_j(t)\sin\theta_j(t), \quad t \in \mathbb{R}, \ j \in \mathbb{Z}_+ \tag{4.3}$$

and constitute an orthonormal basis for $L_r^2(\mathbb{R})$ if and only if there exists an orthonormal basis $\{M_j \in \mathbf{H}^2(\mathbb{C}_+): j \in \mathbb{Z}_+\}$ for $\mathbf{H}^2(\mathbb{C}_+)$ such that

$$M_j(t) = \frac{1}{\sqrt{2}}\rho_j(t)e^{i\theta_j(t)}, \quad t \in \mathbb{R}, \quad j \in \mathbb{Z}_+.$$

In light of the above theorem, two general methods of constructing orthonormal bases for $\mathbf{H}^2(\mathbb{C}_+)$ were proposed in Ref. 24. The first one of them is presented below.

We denote for an $f \in \mathbf{H}^\infty(\mathbb{C}_+)$ by $\mathbf{H}_f^2(\mathbb{C}_+)$ the Hilbert space completed upon the linear space of functions in $\mathbf{H}^2(\mathbb{C}_+)$ under the inner product

$$\langle g, h\rangle_{\mathbf{H}_f^2(\mathbb{C}_+)} := \int_{\mathbb{R}} g(t)\overline{h(t)}|f(t)|^2 dt, \quad g, h \in \mathbf{H}^2(\mathbb{C}_+).$$

Theorem 4.5. *Suppose that $f_1, f_2 \in \mathbf{H}^2(\mathbb{C}_+)$ satisfy that $f_1/f_2 \in \mathbf{H}^\infty(\mathbb{C}_+)$ and f_1 is an outer function. If $e_j \in \mathbf{H}^2(\mathbb{C}_+)$, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_{f_1/f_2}(\mathbb{C}_+)$, then $(f_1/f_2)e_j, j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2(\mathbb{C}_+)$.*

To present the second one, we let the *finite Blaschke product* associated with a finite number of points $z_j \in \mathbb{C}_+$, $j \in \mathbb{N}_n$, be the analytic function f on $\mathbb{C}_+$ defined by

$$f(z) := \prod_{j \in \mathbb{N}_n} \frac{z - z_j}{z - \overline{z_j}}, \quad z \in \mathbb{C}_+.$$

The construction starts with the selection of a sequence of functions $f_n \in \mathbf{H}^\infty(\mathbb{C}_+)$, $n \in \mathbb{N}$, with the properties that $f_n(i) = 0$ and

$$\overline{f_n(t)} = \left(\frac{h_n}{g_n}\right)(t), \quad t \in \mathbb{R}, \ n \in \mathbb{N},$$

where h_n and g_n are analytic functions on $\mathbb{C}_+$ with g_n having at least one but a finite number of zeros in $\mathbb{C}_+$. We then let b_n be the finite Blaschke product associated with the zeros of g_n in $\mathbb{C}_+$, $n \in \mathbb{N}$. Finally, we define

$$\beta_0(z) := \frac{1}{\sqrt{\pi}}\frac{1}{1 - iz}, \ \beta_n(z) := \frac{1}{\sqrt{\pi}}\frac{1}{1 - iz} f_n(z) \prod_{j \in \mathbb{N}_{n-1}} b_j(z), \quad z \in \mathbb{C}_+, \ n \in \mathbb{N}. \tag{4.4}$$

Here we denote $\mathbb{N}_0 := \emptyset$. The following result proved in Ref. 24 ensures that we obtain an orthogonal sequence in $\mathbf{H}^2(\mathbb{C}_+)$.

Theorem 4.6. *The functions β_n, $n \in \mathbb{Z}_+$, constructed by Eq. (4.4) are orthogonal in $\mathbf{H}^2(\mathbb{C}_+)$.*

Completeness of a basis constructed in Theorem 4.6 should be analyzed based on the particular choice of f_n, $n \in \mathbb{N}$.

Two explicit examples of orthonormal bases for $L^2_r(\mathbb{R})$ followed from the two general constructions, Theorems 4.5 and 4.6, were provided in Ref. 24. To introduce the first one, we fix $a \in \mathbb{C}_+$ and $a_r := \Re(a)$, $a_i := \Im(a)$, $b := K(a)$ and

$$\nu := \frac{1}{\sqrt{\pi}} \frac{\sqrt{1-|b|^2}}{|1+\bar{b}|}.$$

Also denote for each $\lambda \in D$ by ζ_λ the real function on $[0, 2\pi]$ defined by

$$\frac{e^{is}-\lambda}{1-\bar{\lambda}e^{is}} = e^{i\zeta_\lambda(s)}, \quad s \in [0, 2\pi].$$

It can be computed that

$$\zeta'_\lambda(s) = \frac{1-|\lambda|^2}{1-2\Re(\lambda e^{-is})+|\lambda|^2}, \quad s \in (0, 2\pi).$$

Setting

$$\rho_j(t) := \frac{\nu}{\sqrt{(t-a_r)^2+a_i^2}}, \quad \theta_j(t) := j\zeta_b(2\arctan t) + \arctan\frac{t-a_r}{a_i}, \quad t \in \mathbb{R}, \quad j \in \mathbb{Z}_+,$$

we obtain that for $t \in \mathbb{R}$, functions

$$\begin{aligned}(\rho_j \cos\theta_j)(t) &= \frac{\nu a_i}{(t-a_r)^2+a_i^2}\cos(j\zeta_b(2\arctan t)) \\ &\quad + \frac{\nu(a_r-t)}{(t-a_r)^2+a_i^2}\sin(j\zeta_b(2\arctan t)),\end{aligned}$$

and

$$\begin{aligned}(\rho_j \sin\theta_j)(t) &= \frac{\nu(t-a_r)}{(t-a_r)^2+a_i^2}\cos(j\zeta_b(2\arctan t)) \\ &\quad + \frac{\nu a_i}{(t-a_r)^2+a_i^2}\sin(j\zeta_b(2\arctan t)),\end{aligned}$$

$j \in \mathbb{Z}_+$, form an orthonormal basis for $L^2_r(\mathbb{R})$ that satisfies (4.3) and $\theta'_j > 0$, $j \in \mathbb{Z}_+$.

To prepare for the second example, we choose pairwise distinct $d_n \in \mathbb{C}_+$, $n \in \mathbb{N}$ that satisfy

$$\sum_{n\in\mathbb{N}} (1-|K(d_n)|) = +\infty.$$

Set $d_{n,r} := \Re(d_n)$, $d_{n,i} := \Im(d_n)$, $b_n := K(d_n)$, $n \in \mathbb{N}$, and

$$\omega_n := \zeta_0 + \sum_{j\in\mathbb{N}_{n-1}} \zeta_{b_j}, \quad n \in \mathbb{N}.$$

The following functions

$$\frac{1}{\sqrt{\pi}}\frac{1}{1+t^2}, \quad \frac{1}{\sqrt{\pi}}\frac{t}{1+t^2}$$

$$\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,i}}{(t-d_{n,r})^2+d_{n,i}^2}\cos(\omega_n(2\arctan t))$$
$$+\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,r}-t}{(t-d_{n,r})^2+d_{n,i}^2}\sin(\omega_n(2\arctan t)), \quad n\in\mathbb{N}$$

and

$$\sqrt{\frac{d_{n,i}}{\pi}}\frac{t-d_{n,r}}{(t-d_{n,r})^2+d_{n,i}^2}\cos(\omega_n(2\arctan t))$$
$$+\sqrt{\frac{d_{n,i}}{\pi}}\frac{d_{n,i}}{(t-d_{n,r})^2+d_{n,i}^2}\sin(\omega_n(2\arctan t)), \quad n\in\mathbb{N},$$

form an orthonormal basis for $L_r^2(\mathbb{R})$. Clearly, the phase of each of the basis functions has a positive derivative.

We remark that Theorems 4.4, 4.5, and 4.6 were extended for $L_{2\pi}^2$ in Ref. 24. Explicit examples satisfying the general constructions can be found, for example, in Refs. 8 and 31. Fast algorithms of decomposing an arbitrary function in $L_{2\pi}^2$ into a sum of the basis functions were developed in Ref. 34.

5. The Bedrosian Identity

The Bedrosian identity is a formula to compute the Hilbert transform of the product of two functions. It plays an important role in the development of HHT and other areas of signal processing.[12] In connection with the development of EMD, there has been significant interest in understanding to what extent the Bedrosian identity holds. Studies on variations and extensions of the Bedrosian identity can be found in Refs. 6, 7, 23, 25, 30–33 and 37–39. In this section, we review several important results in this direction.

The classical Bedrosian identity is

$$[H(fg)](x)=f(x)(Hg)(x), \quad \text{a.e. } x\in\mathbb{R}, \tag{5.1}$$

where $f,g\in L^2(\mathbb{R})$. In Ref. 1, Bedrosian gave a sufficient condition for Eq. (5.1). To state the important result, we need to introduce the Fourier transform $\mathcal{F}$ defined for each $f\in L^2(\mathbb{R})$ at $\xi\in\mathbb{R}$ as

$$\hat{f}(\xi):=(\mathcal{F}f)(\xi):=\int_{\mathbb{R}}f(x)e^{-i\xi x}\,dx.$$

We also denote by $\operatorname{supp} f$ the support of a Lebesgue measurable function f on $\mathbb{R}^d$, $d\in\mathbb{N}$.

Theorem 5.1. *Let* $f, g \in L^2(\mathbb{R})$. *If either* $\operatorname{supp}\hat{f} \subseteq [-a, a]$, $\operatorname{supp}\hat{g} \subseteq (-\infty, -a] \cup [a, \infty)$ *for some* $a \in \mathbb{R}_+ := [0, \infty)$ *or* $\operatorname{supp}\hat{f} \subseteq \mathbb{R}_+$, $\operatorname{supp}\hat{g} \subseteq \mathbb{R}_+$ *then identity* (5.1) *holds.*

The above theorem is known as the Bedrosian theorem and has wide applications in time frequency literature.[9,12,21] Recent mathematical interests in the Bedrosian identity are motivated by Ref. 37, which studied the necessary and sufficient conditions for which the Bedrosian identity is valid. The first characterization for functions that satisfy the Bedrosian identity (5.1) was developed in Ref. 37, which we present below.

Theorem 5.2. *If* $f, f', g \in L^2(\mathbb{R})$ *then the Hilbert transform of function* fg *satisfies the Bedrosian identity* (5.1) *if and only if*

$$\int_{-1}^{0}\int_{\mathbb{R}} \frac{\xi}{t^2} e^{ix\xi(t+1)/t}\hat{f}\left(\frac{\xi}{t}\right)\hat{g}(\xi)d\xi\, dt = 0. \tag{5.2}$$

A sufficient condition was derived from (5.2) in the same paper, which states that if $f, g \in L^2(\mathbb{R})$ are such that

$$\mu(\{t\xi : \xi \in \operatorname{supp}\hat{f},\ t \in [-1, 0]\} \cap \operatorname{supp}\hat{g}) = 0 \tag{5.3}$$

then the Bedrosian identity (5.1) holds, where μ denotes the Lebesgue measure on $\mathbb{R}$. The classical Bedrosian theorem is a special case of this result.

The significance of Theorem 5.2 is that it serves as a base for further study of the Bedrosian identity. Motivated by Theorem 5.2, a new necessary and sufficient condition was proved in Ref. 39.

Theorem 5.3. *If* $f, g \in L^2(\mathbb{R})$ *then the Bedrosian identity* (5.1) *holds if and only if*

$$\int_{\mathbb{R}_+} (\tau_\xi^* \hat{f})(\eta)\hat{g}(-\eta)d\eta = 0, \quad \xi \in \mathbb{R}_+ \tag{5.4}$$

and

$$\int_{\mathbb{R}_-} (\tau_\xi^* \hat{f})(\eta)\hat{g}(-\eta)d\eta = 0, \quad \xi \in \mathbb{R}_- := (-\infty, 0], \tag{5.5}$$

where τ_ξ^* *is the adjoint of the translation operator* τ_ξ *that is defined for each* $\xi \in \mathbb{R}$ *and* $f \in L^2(\mathbb{R})$ *by* $\tau_\xi f := f(\cdot - \xi)$.

It can be seen by Theorem 5.3 that the Bedrosian identity (5.1) is closely related to the closed left translation invariant subspace of $L^2(\mathbb{R}_+)$, that is, the closed subspace $\mathcal{M} \subseteq L^2(\mathbb{R}_+)$ such that $\tau_y^*(\mathcal{M}) \subseteq \mathcal{M}$ for each $y \in \mathbb{R}_+$. By the similarity between (5.4) and (5.5), we use (5.4) for explanation. For each $h \in L^2(\mathbb{R})$ we set

$h^+ := h \cdot \chi_{\mathbb{R}_+}$ and $h^- := h \cdot \chi_{\mathbb{R}_-}$. If $\operatorname{supp} \hat{f} \subseteq [-a, a]$ for some $a \in \mathbb{R}_+$ then

$$\operatorname{span}\{\tau_y^* \hat{f}^+ : y \in \mathbb{R}_+\} \subseteq \mathcal{M}_a := \{h \in L^2(\mathbb{R}_+) : \operatorname{supp} h \subseteq [0, a]\}.$$

The Bedrosian theorem is essentially a consequence of the fact that $\mathcal{M}_a$ is a closed left translation invariant subspace of $L^2(\mathbb{R}_+)$. Studies on other closed left translation invariant subspaces of $L^2(\mathbb{R}_+)$ would yield more sufficient conditions for the Bedrosian identity. In particular, an investigation in Ref. 39 on the condition for $\operatorname{span}\{\tau_y^* \hat{f}^+ : y \in \mathbb{R}_+\}$ to be finite dimensional results in a class of functions f, g satisfying the Bedrosian identity (5.1).

Based on Theorem 5.3, a class of functions $f, g \in L^2(\mathbb{R})$ with explicit expressions that satisfy the Bedrosian identity (5.1) was constructed in Ref. 39. As a consequence, it was observed there that the sufficient conditions in the Bedrosian theorem 5.1 are not necessary for the Bedrosian identity to hold. To see this, we present one pair of f, g in this class. Set

$$f(t) := \frac{1}{\pi(1+t^2)}, \quad \text{and} \quad g(t) := \frac{1}{\pi} \frac{1-2t^2}{4+5t^2+t^4}, \quad t \in \mathbb{R}.$$

The Fourier transforms of f and g are given by

$$\hat{f}(\xi) = \exp(-|\xi|) \quad \text{and} \quad \hat{g}(\xi) = \exp(-|\xi|) - \frac{3}{2}\exp(-2|\xi|), \quad \xi \in \mathbb{R}.$$

It can be verified directly that Eqs. (5.4) and (5.5) are satisfied. Therefore, f and g given above satisfy the Bedrosian identity (5.1) while have the property that $\operatorname{supp} \hat{f} = \operatorname{supp} \hat{g} = \mathbb{R}$.

Surprisingly, the following necessity of the Bedrosian theorem was obtained in Ref. 38.

Theorem 5.4. *If $f, g \in L^2(\mathbb{R})$ satisfy the Bedrosian identity* (5.1), $\operatorname{supp} \hat{f} \subseteq [-a, b]$ *for some $a, b \in \mathbb{R}_+$ and endpoints $-a, b$ are in* $\operatorname{supp} \hat{f}$ *then* $\operatorname{supp} \hat{g} \subseteq \mathbb{R} \backslash [-b, a]$.

The above theorem might be interpreted as that if $f \in L^2(\mathbb{R})$ is of low Fourier frequencies then for the Bedrosian identity (5.1) to hold, it is necessary and sufficient that g has high Fourier frequencies. Theorem 5.4 was first proved in Ref. 39 under the additional assumption that $\hat{f} \cdot \chi_{[-a,b]}$ is the restriction on $[-a, b]$ of a nontrivial real-analytic function. Another necessity of the Bedrosian theorem was observed in Ref. 33. Specifically, it was shown there that a bounded linear translation invariant operator on $L^2(\mathbb{R}^d)$ satisfies the Bedrosian theorem if and only if it is a linear combination of the identity operator and the partial Hilbert theorems. In the one-dimensional case, it states that the Hilbert transform is essentially the only bounded linear translation invariant operator on $L^2(\mathbb{R})$ that satisfies the Bedrosian theorem.

Finally, we mention some recent work on the Bedrosian identity for L^p functions.[25,31,38] The Bedrosian theorem for L^p functions was established in

Refs. 25 and 38. The characterization in Theorem 5.3 was extended to L^p functions in Ref. 38. Reference 31 obtained another characterization in the time domain.

6. Conclusion

EMD and the classical Fourier analysis are two different methods for data analysis. There is a large gap between them. EMD is very adaptive to the data under consideration but it lacks mathematical justification. On the other hand, the classical Fourier analysis has a rigorous mathematical foundation, but it is a linear process and is not adaptive to the data under consideration. Recent mathematical developments on EMD focused on bridging the gap between EMD and the classical Fourier analysis. There were two major directions. The first direction was to modify the EMD so that it is less "empirical" and more "mathematical". The second direction was to modify the classical Fourier analysis so that it is more adaptive and more nonlinear. Research results obtained so far in this area are interesting and insightful, although there is much more ahead to be done.

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