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ADAPTIVE FREQUENCY ESTIMATION WITH LOW SAMPLING RATES BASED ON ROBUST CHINESE REMAINDER THEOREM AND IIR NOTCH FILTER

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In this paper, based on an adaptive IIR notch filter and a robust Chinese remainder theorem (CRT), we propose an adaptive frequency estimation algorithm from multiple undersampled sinusoidal signals. Our proposed algorithm can significantly reduce the sampling rates and provide more accurate estimates than the method based on adaptive IIR notch filter and sampling rates above the Nyquist rates does. We then present simulation results to verify the performance of our proposed algorithm for both stationary and nonstationary signals.

Keywords: Adaptive IIR notch filter; Chinese remainder theorem (CRT); frequency estimation; robust CRT; undersampling.

1. Introduction

Frequency estimation of a sinusoidal signal contaminated in an additive noise is a classical problem and has many practical applications in, for example, power, radar, sonar, communication, and biomedical engineering systems. There are two classes of frequency estimation algorithms. One class is mainly for off-line processing, which is based on power spectral estimation techniques including traditional periodogram and super-resolution algorithms such as Capon and MUSIC algorithms.[1,2] These methods can be applied to achieve more accurate frequency estimations but usually require a higher computational cost. The second class is for on-line processing and applies to frequency estimation and tracking for nonstationary signals. This class of on-line methods consists of two types. One type is adaptive FIR filters that usually require high filter orders, i.e. large number of coefficients, to obtain a satisfactory sharp cutoff frequency. The other type is adaptive IIR filters,[3,4] adaptive IIR notch filters, in particular. An adaptive IIR notch filter is a filter whose magnitude response vanishes at a particular value (the notch frequency f_0) on the unit circle, and whose magnitude response is nearly constant at other points on the unit circle. A good notch IIR filter can be obtained by only using a second-order approximation. In the meantime, a complementary bandpass filter may be realized by subtracting a notch filter output from its input, which may yield a sharp cut-off frequency and allow the retrieval of a sinusoidal component of the input while the background noise is significantly reduced. Adaptive IIR notch filters have wide applications, in, for example, sonar.[3,4] One of the adaptive IIR notch filter designs is to use lattice structure,[3–6] where only a minimum number of filter coefficients are needed to be adapted and it allows independent tunings of the notch frequency and the attenuation bandwidth compared with a direct form structure.

It is well-known that frequencies can be uniquely determined from a sampled signal when the sampling rate is above the Nyquist rate. If the sampling rate is below the Nyquist rate, it is in general not possible to uniquely determine the frequencies from a single sampled waveform. If there are multiple sampled copies of a single signal with different sampling rates, then the frequencies may be uniquely determined even all the sampling rates are below the Nyquist rate[7,8] by using (generalized) Chinese remainder theorem (CRT). The basic idea is that with the undersampled waveforms, folded frequencies (remainders) can be detected and the CRT can be used to determine the true frequencies from the detected folded frequencies/remainders. However, it is well known that the conventional CRT is not robust in the sense that a small error in its remainders may cause a large error in the solution and it is often that the remainders have errors due to the noisy and nonstationary environment in practice that is particularly the case when an adaptive IIR notch filter is used. This means that the conventional CRT may not provide a desired solution in practice. Recently, we have developed a robust CRT and the robust CRT provides a robust frequency estimation[10] when additive noise is concerned in a sinusoidal signal and the DFT is used to determine the folded frequencies/remainders.

In this paper, we propose an adaptive frequency estimation algorithm from multiple undersampled waveforms. In this algorithm, we first apply several adaptive IIR notch filters to estimate the folded frequencies (remainders) from several sampled copies of a single signal waveform with different undersampling rates. We then estimate the signal frequency using the robust CRT from these detected folded frequencies. Note that the estimated frequencies from sampled signals in the discrete-time domain are normalized frequencies in the range $[0, 0.5]$ and then the true frequencies of the analog signals are the product of the estimated normalized frequencies with the sampling frequencies. Since the estimated normalized frequencies are most-likely not accurate in practice, the errors may be magnified by the sampling frequencies. Therefore, when the sampling frequencies are reduced, the accuracy of the estimated analog frequencies is increased. This is another advantage of our proposed algorithm with low sampling rates. We then present simulation results to illustrate the effectiveness of the proposed adaptive algorithm for both stationary and non-stationary signals. Our simulation results show that our proposed algorithm cannot only reduce the sampling rates significantly but also improve the analog frequency estimation accuracy significantly compared to the one using the adaptive notch filter with sampling rate above the Nyquist rate.

The remaining of this paper is organized as follows. In Sec. 2, we first briefly describe the problem, introduce the frequency estimation algorithm using adaptive IIR lattice notch filter and the robust CRT, and then present our adaptive frequency estimation algorithm from undersampled waveforms. In Sec. 3, we present some simulation results. In Sec. 4, we conclude this paper.

2. Adaptive Frequency Estimation with Low Sampling Rates

Let us first describe the problem.

2.1. *Problem description*

Consider a noisy sinusoidal signal with an unknown frequency f_0 and amplitude A:

$$x(t) = A\cos(2\pi f_0 t + \theta) + w(t), \tag{1}$$

where θ is the phase that is uniformly distributed between 0 and 2π and $w(t)$ is an additive (white or colored) noise. Its sampled waveform $x(n)$ with sampling rate f_{s} is:

$$x(n) = A\cos(2\pi n f_0/f_{\mathrm{s}} + \theta) + w(n/f_{\mathrm{s}}). \tag{2}$$

In order to uniquely determine the frequency, the sampling frequency f_{s} has to be at least as twice large as the signal frequency, i.e. above the Nyquist rate. If the signal frequency f_0 is high, the sampling frequency f_{s} has to be high. However, in many practical applications, low sampling frequencies are of great interest: (i) low sampling frequencies may reduce the hardware cost; (ii) in some cases only low sampling frequencies are available; (iii) the signal frequencies of interest may change (unknownly) in an nonstationary environment while the sampling frequency is fixed; (iv) as mentioned in Sec. 1, a lower sampling frequency leads to a less error in an analog frequency estimate from a normalized frequency estimate of sampled discrete-time signals.

Although for a single sampled signal with a low sampling rate (lower than the Nyquist rate called undersampling), it is not possible to uniquely determine the signal frequency, we next propose to use multiple sampled copies of a single signal with multiple low sampling rates $M_1, M_2, \ldots, M_L$ in Hertz

$$x_i(n) = A_i\cos(2\pi n f_0/M_i + \theta) + w(n/M_i), \tag{3}$$

for $i = 1, 2, \ldots, L$, where $M_i \ll f_0$. In this case, for each i, $1 \le i \le L$, the folded frequency $f_{0,i} = f_0 - n_i M_i$ for an integer n_i such that $0 \le f_{0,i} < M_i$ can only be estimated from the ith undersampled signal copy $x_i(n)$. For each sampled signal copy $x_i(n)$, we next propose to use an adaptive lattice IIR notch filter to estimate the frequency (folded frequency in this undersampling case).

2.2. *Adaptive lattice IIR notch filter*

As we mentioned in Sec. 1, due to the low computational cost and the adaptivity for nonstationary signals, we next consider and introduce adaptive IIR notch filters. The transfer function $H(z)$ of an adaptive IIR notch filter with lattice structure can be expressed as[3]:

$$H(z) = \frac{1+\sin\theta_2}{2}\frac{1+2\sin\theta_1 z + z^2}{1+2\sin\theta_1(1+\sin\theta_2)z+\sin\theta_2 z^2}, \tag{4}$$

where

$$\theta_1 = 2\pi f_0/f_{\mathrm{s}} - \pi/2, \tag{5}$$

$$\sin\theta_2 = \frac{1-\tan(B/2)}{1+\tan(B/2)}, \quad 0<\theta_2<\pi/2, \tag{6}$$

where f_0 is the notch frequency and B is the 3 dB attenuation bandwidth. From the above properties for the parameters θ_1 and θ_2, the independent tunings of the notch frequency and attenuation bandwidth are possible. When the attenuation bandwidth parameter θ_2 is under-held constant that is given and some initializations of $x_1(0)$, $x_2(0)$ and $\theta_1(0)$ are given, we can use the following formulas to tune the notch frequency parameter θ_1. When $x_1(n)$ and $x_2(n)$ are calculated and $x(n)$ is the input data, g_1 and g_2 can be calculated via Eq. (7) below, $\mu(n)$ is calculated via Eq. (11), and $y(n)$ is calculated via Eq. (8). With $\theta_1(n)$ and $\mu(n)$, $\theta_1(n+1)$ can be calculated via Eq. (9) and then, $x_1(n+1)$ and $x_2(n+1)$ can be calculated via Eq. (10), which turns to next time $n+1$.

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} x(n) \\ x_2(n) \end{bmatrix}, \tag{7}$$

$$y(n) = (1/2)[x(n)+g_2], \tag{8}$$

$$\theta_1(n+1) = \theta_1(n) - \mu(n)y(n)x_1(n), \tag{9}$$

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1(n+1)) & -\sin(\theta_1(n+1)) \\ \sin(\theta_1(n+1)) & \cos(\theta_1(n+1)) \end{bmatrix} \begin{bmatrix} g_1 \\ x_1(n) \end{bmatrix}, \tag{10}$$

$$\mu(n) = \frac{\mu_0}{\sum_{k=0}^{n}\lambda^{n-k}[x_1(k)]^2}, \quad 0 \ll \lambda < 1, \tag{11}$$

where $x(n)$ and $y(n)$ are the input and the output of the adaptive filter, respectively, $\mu(n)$ is a variable step-size parameter, $\mu_0>0$ is a given initial step-size, and λ is a forgetting factor. For the stability of the algorithm at lower SNR, we propose to use the following steady variable step-size:

$$\mu(n) = \frac{\mu_0}{c+\sum_{k=0}^{n}\lambda^{n-k}[x_1(k)]^2}, \quad 0 \ll \lambda < 1, \tag{12}$$

where $c > 0$ is a factor controls the steady of algorithm at low SNR. With the adaptively calculated $\theta_1(n)$ above, the final estimate of f_0 is

$$\hat{f}_0 = \hat{f}_0(n) = \frac{(\theta_1(n) + \pi/2)f_{\mathrm{s}}}{2\pi} \tag{13}$$

at the nth step. From the above adaptive frequency estimation, one can see that an estimation error in $\theta_1(n)$ from the adaptive IIR notch filter will be magnified by the sampling frequency f_{s}. Thus, a smaller sampling frequency may lead to a lower analog frequency estimation error.

2.3. *Robust CRT*

We now describe the robust CRT we recently obtained in Ref. 10 and the following robust CRT is self-contained. Let N be a positive integer that corresponds to f_0 in the above frequency estimation problem, $0 < M_1 < M_2 < \cdots < M_L$ be the L moduli that correspond to the sampling rates, and $r_1, r_2, \ldots, r_L$ be the L remainders of N, i.e.

$$N \equiv r_i \mod M_i \quad \text{or} \quad N = n_i M_i + r_i, \tag{14}$$

where $0 \le r_i < M_i$ and n_i is an unknown integer, for $1 \le i \le L$. It is known that N can be uniquely reconstructed from its L remainders r_i if and only if $0 \le N <$ lcm $(M_1, M_2, \ldots, M_L)$, where lcm stands for the least common multiple. The problem here is how to robustly reconstruct N when the remainders r_i have errors:

$$0 \le \hat{r}_i \le M_i - 1 \quad \text{and} \quad |\hat{r}_i - r_i| \le \tau, \tag{15}$$

where $\tau < \min_{1 \le i \le L} M_i$ is the maximal error level, called remainder error bound and the erroneous remainders $\hat{r}_i$ correspond to the folded frequency estimates $\hat{f}_0$ in Eq. (13) from the ith undersampled signal copy $x_i(n)$ for each i with $1 \le i \le L$. We now want to reconstruct N from these erroneous remainders $\hat{r}_i$ and the known moduli M_i. With these erroneous remainders, Eq. (14) becomes

$$N = n_i M_i + \hat{r}_i + \triangle r_i, \quad 1 \le i \le L, \tag{16}$$

where n_i are unknown and $\triangle r_i = r_i - \hat{r}_i$ denote the errors of the remainders. From Eq. (15), $|\triangle r_i| \le \tau$. The basic idea for our robust CRT is to accurately determine the unknown integers n_i in Eq. (16) which are the folding integers that may cause large errors in the reconstructions if they are erroneous. The robust CRT we recently obtained in Ref. 10 is as follows.

Let M denote the greatest common divisor (gcd) of all the moduli M_i. Then,

$$M_i = M\Gamma_i, \quad 1 \le i \le L, \tag{17}$$

and all Γ_i, $1 \le i \le L$, are co-prime, i.e. the gcd of any pair Γ_i and Γ_j for $i \ne j$ is 1.

For $1 \le i \le L$, let

$$\gamma_i \triangleq \Gamma_1 \cdots \Gamma_{i-1}\Gamma_{i+1} \cdots \Gamma_L, \tag{18}$$

where $\gamma_1 \triangleq \Gamma_2 \cdots \Gamma_L$ and $\gamma_L \triangleq \Gamma_1 \cdots \Gamma_{L-1}$. Since $M_1 < M_2 < \cdots < M_L$, we have $\Gamma_1 < \Gamma_2 < \cdots < \Gamma_L$.

For each i with $2 \leq i \leq L$, define

$$S_i \triangleq \left\{(\bar{n}_1, \bar{n}_i) = \operatorname{argmin}_{\substack{\hat{n}_1=0,1,\ldots,\gamma_1-1 \\ \hat{n}_i=0,1,\ldots,\gamma_i-1}} |\hat{n}_i M_i + \hat{r}_i - \hat{n}_1 M_1 - \hat{r}_1|\right\}, \tag{19}$$

and let $S_{i,1}$ denote the set of all the first components $\bar{n}_1$ of the pairs $(\bar{n}_1, \bar{n}_i)$ in set S_i, i.e.

$$S_{i,1} \triangleq \{\bar{n}_1 \mid (\bar{n}_1, \bar{n}_i) \in S_i \text{ for some } \bar{n}_i\} \tag{20}$$

and define

$$S \triangleq \bigcap_{i=2}^{L} S_{i,1}. \tag{21}$$

Then, we have the following result.

Theorem 1. *If*

$$0 \leq N < \operatorname{lcm}(M_1, M_2, \ldots, M_L) = M\Gamma_1\Gamma_2 \cdots \Gamma_L \tag{22}$$

and

$$M > 4\tau \tag{23}$$

then, set S defined above contains only element n_1, i.e. $S = \{n_1\}$, and furthermore if $(n_1, \bar{n}_i) \in S_i$, then $\bar{n}_i = n_i$ for $2 \leq i \leq L$, where n_i, $1 \leq i \leq L$, are the true solutions in Eq. (16).

For the completeness, its proof is in Appendix. When the folding integers n_i in Eq. (16) are accurately solved, the unknown N can be estimated as

$$\hat{N} = \left[\frac{1}{L}\sum_{i=1}^{L}(n_i M_i + \hat{r}_i)\right] \tag{24}$$

where $[\cdot]$ stands for the rounding integer (rounding to the closest integer) and the estimate error is thus upper bounded by

$$|N - \hat{N}| \leq \tau. \tag{25}$$

The above estimate error of N is due to the remainder errors $r_i - \hat{r}_i$ that has the maximal level τ. One can clearly see that this reconstruction is robust when the gcd of all the moduli is not 1 and thus called robust CRT. Although the above robust CRT is based on 2-D searchings in Eq. (19), a fast algorithm based on 1-D searchings has been obtained in Ref. 10.

2.4. *Adaptive frequency estimation from undersampled waveforms using robust CRT and adaptive notch filter*

After the above adaptive IIR notch filter and robust CRT are described, our proposed adaptive frequency estimation from undersampled waveforms is quite simple. For each of the multiple sampled copies $x_i(n) = x(n/M_i)$ with undersampling rates $f_{si} = M_i$ in Eq. (3), the adaptive IIR notch filter described in Sec. 2.2 is applied and an estimate $\hat{r}_i = \hat{f}_{0,i}(n)$ in Eq. (13) at stage n is:

$$\hat{r}_i = \frac{(\theta_1(n) + \pi/2) f_{si}}{2\pi} =: \frac{\arccos(\cos(\theta_1(n) + \pi/2))}{2\pi} M_i, \tag{26}$$

for $1 \le i \le L$. From these estimated folded frequencies/remainders $\hat{r}_i$, $1 \le i \le L$, the robust CRT described in Sec. 2.3 is applied to get an estimate $\hat{N}$ (corresponds to $\hat{f}_0$) of the integer N (corresponds to f_0). Note that the arccos(cos) function used in Eq. (26) is to ensure the correct range of an estimate from the adaptive algorithm. Interestingly, although the remainders $\hat{r}_i$ used in the robust CRT in Sec. 2.3 are integers, the robust CRT works and Theorem 1 still holds when the estimated folded frequencies $\hat{r}_i$ in Eq. (26) are reals since they can be plugged into Eq. (19) to find the sets S_i no matter they are integers or reals. When $\hat{r}_i$ are not integers, the estimate $\hat{N}$ in Eq. (24) does not need to be rounded to an integer and thus becomes

$$\hat{f}_0 = \hat{N} = \frac{1}{L} \sum_{i=1}^{L} (n_i M_i + \hat{r}_i), \tag{27}$$

after the folding integers n_i are determined in Theorem 1.

A remark is that the major difference of the frequency estimation method proposed between this paper and Ref. 10 is that the proposed method in this paper is on-line and adaptive, while the one in Ref. 10 is not. Our proposed adaptive frequency estimation from multiple undersampled waveforms is illustrated in Fig. 1.

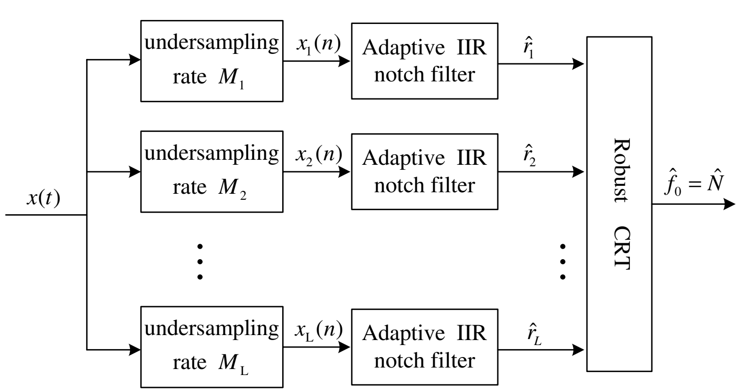


Fig. 1. Adaptive frequency estimation with multiple undersampled waveforms.

3. Simulation Results

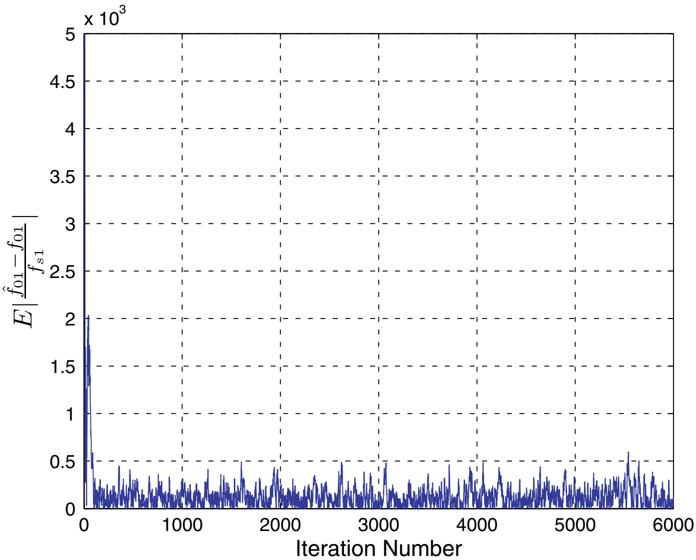
In all the simulations in this section, the parameters in the adaptive IIR notch filters are: $x_1(0) = x_2(0) = 0$, $\mu_0 = 0.03$, $\theta_2 = 0.45\pi$, $\theta_1(0) = 1.4\pi$, $c = 2$ and $\lambda = 0.98$ in Eq. (12). The number of signal samples used is always 6000 and the additive noise is white Gaussian. Each result is based on 1000 Monte-Carlo trials.

We consider two examples. In Example 1, we consider an unknown but fixed frequency, $f_0 = N = 120\,\text{KHz}$. We compare three cases: Case (i) adaptive IIR notch filters and robust CRT with multiple undersampling rates; Case (ii) adaptive IIR notch filter with sampling rate above the Nyquist rate; Case (iii) adaptive IIR notch filters and the conventional CRT with multiple undersampling rates. In Case (i), two sampling rates $M_1 = 17\,M = 17{,}000\,\text{Hz}$ and $M_2 = 19\,M = 19{,}000\,\text{Hz}$ with $M = 1000$. Clearly frequency f_0 falls in the range Eq. (22) in the robust CRT. In Case (ii), the sampling rate $f_{\text{s}} = 10 f_0 = 1.2\,\text{MHz}$. In Case (iii), the two sampling rates are $M_1 = 16{,}993\,\text{Hz}$ and $M_2 = 1{,}9001\,\text{Hz}$ that are close to the two sampling rates used in Case (i) but they are co-prime in order to use the conventional CRT. One can see that the sampling rates in Case (i) are over 6 times less than the signal frequency.

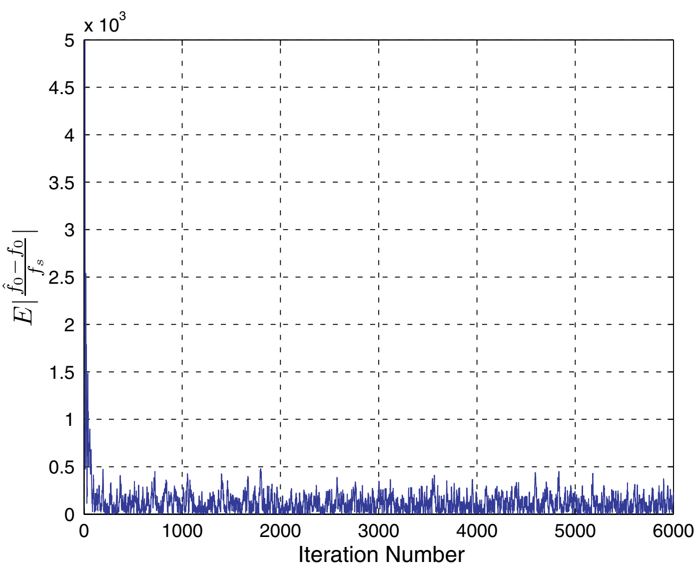
The SNR for Figs. 2 and 3 is 0 dB. Figure 2(a) shows the mean normalized error $E(|\hat{f}_{0,1} - f_{0,1}|/f_{s1})$ for the first undersampling in Case (i) and Fig. 2(b) shows the mean normalized error $E(|\hat{f}_0 - f_0|/f_{\text{s}})$ for Case (ii) with sampling rate above the Nyquist rate and both use the adaptive IIR notch filtering. One can see from these two figures that the normalized frequency errors are in the same level and thus a smaller sampling frequency leads to a smaller frequency estimation error in the analog frequency f_0, which can be seen from Fig. 3 too.

Figure 3(a) shows the mean error of the estimated analog frequency $\hat{f}_0$ using the robust CRT and the two estimated folded frequencies from the adaptive IIR notch filters with the two undersampled rates in Case (i). Figure 3(b) shows the mean error of the estimated analog frequency $\hat{f}_0$ using the adaptive IIR notch filter with sampling rate above the Nyquist rate in Case (ii). One can see that there is an order of magnitude difference in the two cases and the undersampled case is much better.

For Case (iii), the estimated folded frequencies using adaptive IIR notch filter is shown in Fig. 4(a) and then the estimated analog frequency using the conventional CRT is shown in Fig. 4(b). In this case, the SNR is 20 dB and two true-folded frequencies are 1049 and 5994 Hz for the undersampling rates $M_1 = 16{,}993\,\text{Hz}$ and $M_2 = 19{,}001\,\text{Hz}$, respectively. One can see that although the two folded frequencies are estimated well (with much smaller errors than the ones in Case (i) due to the much higher SNR) and the sampling rates are similar to the sampling rates used in Case (i), the conventional CRT fails completely due to its non-robustness. This confirms that our proposed robust CRT indeed provides a robust frequency estimation solution.

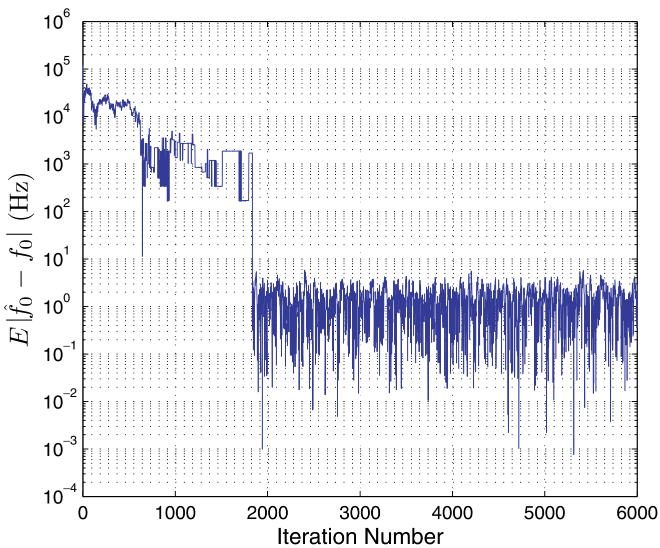


(a)

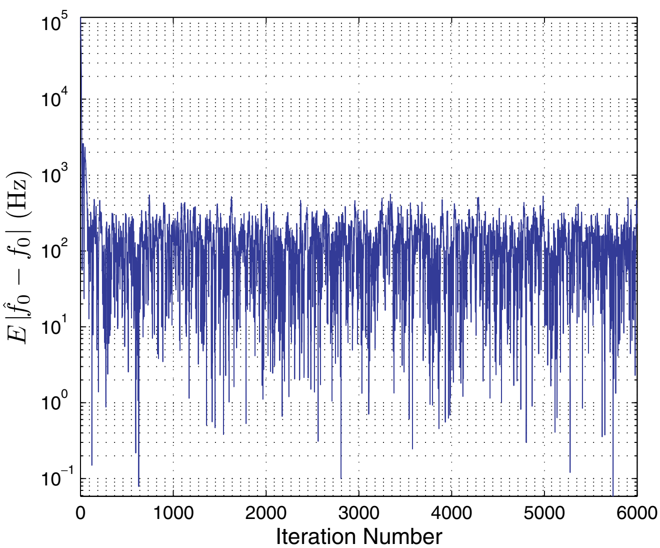


(b)

Fig. 2. (a) Mean error of the estimated normalized frequency from the adaptive IIR notch filter with the first undersampling rate. (b) Mean error of the estimated normalized frequency from the adaptive IIR notch filter with sampling rate above the Nyquist rate.

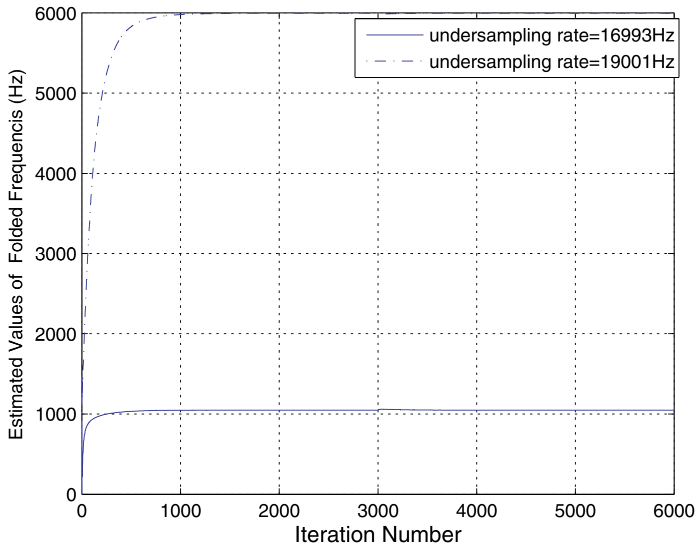


(a)

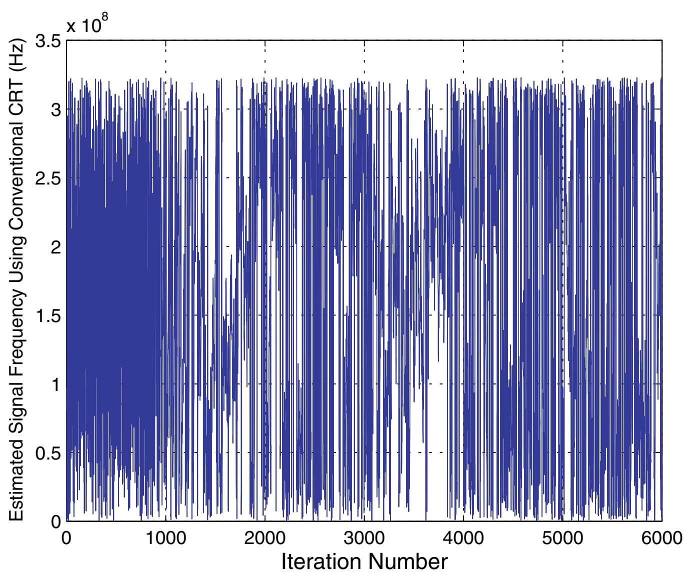


(b)

Fig. 3. (a) Mean error of the estimated analog frequency $\hat{f}_0$ from the adaptive IIR notch filter and the robust CRT with the two undersampling rates in Case (i). (b) Mean error of the estimated analog frequency $\hat{f}_0$ from the adaptive IIR notch filter with sampling rate above the Nyquist rate in Case (ii).



(a)



(b)

Fig. 4. (a) Estimated folded frequencies $\hat{r}_1 = \hat{f}_{0,1}$ and $\hat{r}_2 = \hat{f}_{0,2}$ from the adaptive IIR notch filter with the two undersampling rates in Case (iii). (b) The estimated analog frequency $\hat{f}_0$ from the two estimated folded frequencies in (a) using the conventional CRT.

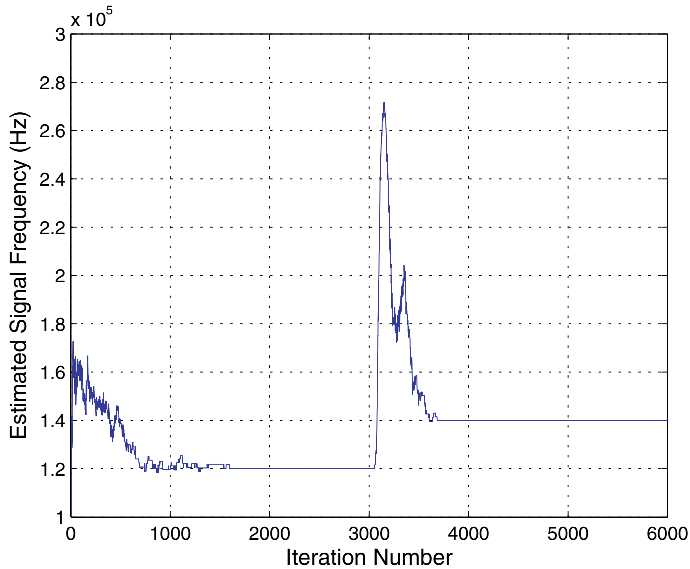


Fig. 5. Estimated time-varying frequency using the robust CRT, where two true frequencies are 120 and 140 KHz.

In Example 2, we consider a nonstationary signal and the noisy discrete-time sinusoidal signal with time-varying frequency is:

$$x(n) = \begin{cases} A\cos(2\pi n f_0/M_i + \theta) + w(n/M_i), & 0 \le n \le 3000 \\ A\cos(2\pi n f_1/M_i + \theta) + w(n/M_i), & 3001 \le n \le 6000 \end{cases} \tag{28}$$

where $f_0 = 120\,\text{KHz}$ and $f_1 = 140\,\text{KHz}$ and the SNR is $0\,\text{dB}$.

This simulation is to illustrate the adaptivity of our proposed frequency estimation algorithm. We still use two undersamplings and the two sampling frequencies are $M_1 = 17\,M = 17{,}000\,\text{Hz}$ and $M_2 = 19\,M = 19{,}000\,\text{Hz}$ with $M = 1000$. Clearly frequencies f_0 and f_1 are all in the range (22). Figure 5 shows that the proposed algorithm can well track the true frequencies in the range $[0, M\Gamma_1\Gamma_2\cdots\Gamma_L)$.

4. Conclusion

In this paper, we have proposed an adaptive frequency estimation algorithm using adaptive IIR notch filter and the robust Chinese remainder theorem (robust CRT) when multiple undersampled signals are used. We have shown that our proposed algorithm not only significantly reduces the sampling rates but also significantly improves the frequency estimation accuracy. We have also shown that our proposed algorithm can track the time-varying frequency well in a nonstationary environment.

Appendix: Proof of Theorem 1

Proof. If the conditions in Theorem 1 are satisfied, it is not hard to see that the true solution n_i in Eq. (16) falls in the range $0 \leq n_i < \gamma_i$ for $1 \leq i \leq L$. Thus, for any pair $(\bar{n}_1, \bar{n}_i) \in S_i$ for $2 \leq i \leq L$, we have

$$|\bar{n}_i M_i + \hat{r}_i - \bar{n}_1 M_1 - \hat{r}_1| \leq |n_i M_i + \hat{r}_i - n_1 M_1 - \hat{r}_1|. \tag{29}$$

Let $\mu_i = \bar{n}_i - n_i$ for $1 \leq i \leq L$, and replace $\hat{r}_i$ by $N - n_i M_i - \Delta r_i$ in both sides of Eq. (29) and we then have

$$|\mu_i M_i - \mu_1 M_1 - (\Delta r_i - \Delta r_1)| \leq |\Delta r_i - \Delta r_1|. \tag{30}$$

Therefore, according to Eqs. (15) and (23), we have

$$|\mu_i M_i - \mu_1 M_1| \leq 2|\Delta r_i - \Delta r_1| \leq 2(|\Delta r_i| + |\Delta r_1|) \leq 4\tau < M. \tag{31}$$

Dividing M in both sides of Eq. (31), we have

$$|\mu_i \Gamma_i - \mu_1 \Gamma_1| < 1. \tag{32}$$

Since μ_i, Γ_i, μ_1, and Γ_1 are all integers, Eq. (32) implies

$$\mu_i \Gamma_i = \mu_1 \Gamma_1, \quad \text{for } i = 2, 3, \ldots, L. \tag{33}$$

Since Γ_i and Γ_1 are co-prime for $2 \leq i \leq L$, we have

$$\mu_1 = h\Gamma_i \quad \text{and} \quad \mu_i = h\Gamma_1, \text{ i.e. } \bar{n}_1 = n_1 + h\Gamma_i \quad \text{and} \quad \bar{n}_i = n_i + h\Gamma_1 \tag{34}$$

for integer h with $|h| < \min(\gamma_i, \gamma_1)$. Substituting Eq. (34) into Eq. (29), we obtain

$$|\bar{n}_i M_i + \hat{r}_i - \bar{n}_1 M_1 - \hat{r}_1| = |n_i M_i + \hat{r}_i - n_1 M_1 - \hat{r}_1|, \tag{35}$$

which implies $(n_1, n_i) \in S_i$ for $i = 2, 3, \ldots, L$. This proves $n_1 \in S$. We next show $S = \{n_1\}$. Property (34) also implies

$$S_i = \{(n_1 + h\Gamma_i, n_i + h\Gamma_1) : \text{ for integers } h \text{ with } |h| < \min(\gamma_i, \gamma_1)\}. \tag{36}$$

If $\bar{n}_1 \in S$, then $\bar{n}_1 \in S_{i,1}$ for $i = 2, 3, \ldots, L$, and therefore, according to the definition of $S_{i,1}$ in Eqs. (19) and (36), we have $\bar{n}_1 - n_1 = h\Gamma_i$ for some integer h with $|h| < \min(\gamma_i, \gamma_1)$ for $i = 2, 3, \ldots, L$. This implies that $\bar{n}_1 - n_1$ divides all Γ_i for $i = 2, 3, \ldots, L$, and thus is a multiple of the product of Γ_i, $i = 2, 3, \ldots, L$, i.e. a multiple of γ_1. Since $0 \leq \bar{n}_1, n_1 \leq \gamma_1 - 1$, we can conclude $\bar{n}_1 - n_1 = 0$. This proves that $S = \{n_1\}$. In the meantime, $\bar{n}_1 = n_1$ implies $h = 0$ in Eq. (36), i.e. $\bar{n}_i = n_i$ for $i = 2, 3, \ldots, L$. Hence, Theorem 1 is proved. □

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