

ESTIMATION OF INSTANTANEOUS FREQUENCY PARAMETERS OF THE OPERATOR-BASED SIGNAL SEPARATION METHOD

XIYUAN HU*, SILONG PENG* and WEN-LIANG HWANG†

**Institute of Automation, The Chinese Academy of Sciences
Beijing, China*

†Institute of Information Science, Academia Sinica, Taiwan

The main function of the operator-based signal separation approach is to construct an operator from a signal and use it to decompose the signal into two subcomponents. The procedure involves two steps: estimating the operator, and decomposing the signal into two subcomponents. Existing approaches estimate the operator's parameters from the local extrema of a signal. In contrast, we show that the parameters can be estimated by adopting a variational approach. Because the proposed approach imposes a global constraint on the operator's parameter, the estimated parameters are more robust than those derived from the local extrema of the signal. We also compare the signal separation results with those obtained by using the empirical mode decomposition (EMD) method.

Keywords: Operator-based signal processing; instantaneous frequency; variational approach; empirical mode decomposition (EMD).

1. Introduction

Signal decomposition has been applied in many fields. The most widely used approach models a signal as a superposition of additive basic signals. For example, the basic component of Huang *et al.*'s empirical mode decomposition (EMD) algorithm is the intrinsic mode function (IMF).⁷ To separate signals, the operator-based signal separation approach uses an adaptive singular local linear operator. A singular local linear operator is a singular linear operator whose support is compact. If a singular local linear operator is applied to a signal, the part of the signal in the null space of the operator will be removed by the operator; and, the resultant signal will not contain any information that is in the null space of the operator. In other words, applying the operator to a signal decomposes the signal into two subcomponents, one is in the operator's null space and the other is not. Implementing the operator-based approach usually involves two steps: estimating the operator's parameters, and using the estimated operator to extract the null space component of the signal in a variational formula.

To decompose a signal, Peng and Hwang⁸ has proposed using several operators to extract the local narrow band subcomponents of the signal. In a local narrow

band signal, the local interval of the signal at any point can be approximated as $A(t) \cos(\omega t + \phi(t))$, where $A(t)$ is a band-limited signal whose maximal frequency is much smaller than ω , and $\phi(t)$ is a slow-varying phase function. An operator can be derived by applying differentiation or integration techniques. For example, the local narrow band signal $\cos(\varpi(t_0)t + c)$ with t in the neighborhood of t_0 is in the null space of the operator $(d^2/dt^2) + \varpi^2(t_0)$:

$$\left(\frac{d^2}{dt^2} + \varpi^2(t_0) \right) \cos(\varpi(t_0)t + c) = 0. \quad (1)$$

Note that the operator parameter $\varpi(t_0)$ is the instantaneous frequency of the signal at point t_0 and is to be estimated from the signal. A number of methods can be used to derive the instantaneous frequency.^{3,4,6} In Ref. 8, it is estimated from the local extrema of a signal. However, we found that this method can be unreliable if the signal is noisy because of spurious extrema caused by the noise. To resolve the problem, we propose using a variational approach, which impose constraints on the estimated instantaneous frequencies; as a consequence, it can derive reliable instantaneous frequency estimation results. A similar approach for estimating instantaneous frequencies in the time–frequency representation of a noisy signal is proposed in Ref. 5.

In this paper, we use the variational approach to estimate the instantaneous frequency parameters of an operator as well as the null space component of a signal. We compare the performance of the proposed signal separation method with that in Ref. 8. In addition, we compare our results with those derived by the EMD method.

The remainder of the paper is as follows. In Sec. 2, we review the operator-based signal separation approach. In Sec. 3, we present our variational approach for instantaneous frequency estimation. In Sec. 4, we describe simulations conducted using both synthetic and practical examples. Section 5 contains some concluding remarks.

2. Operator-Based Signal Separation

Operator-based signal separation decomposes a signal $S(t)$ into additive sub-components as follows:

$$S(t) = \sum_{i=1}^M S_i(t) + R^M S(t). \quad (2)$$

This can be achieved iteratively by first decomposing the signal $S(t)$ into the sum of $S_1(t)$ and $R^1 S(t)$ by an operator \mathcal{T}_1 derived from $S(t)$. Then, $R^1 S(t)$ is decomposed by an operator \mathcal{T}_2 derived from $R^1 S(t)$, which yields $S_2(t)$ and $R^2 S(t)$. Subsequent iterations are performed in the same way. Let $R^0 S(t) = S(t)$. To obtain $R^i S(t)$ for $i = 1, \dots, M$, we solve the following optimization problem:

$$R^i S(t) = \arg \min_U \{ \|\mathcal{T}_i(R^{i-1} S - U)\|^2 + \lambda_i \|D(U)\|^2 \}, \quad (3)$$

where \mathcal{T}_i is the operator derived from $R^{i-1}S(t)$, D is the identity matrix, which regulates the residual signal U , and λ_i is a Lagrangian parameter. Minimizing the term $\|\mathcal{T}_i(R^{i-1}S - U)\|^2$ indicates that $R^{i-1}S - U$ is in the null space of \mathcal{T}_i . The analytical solution of (3) is

$$R^i S(t) = (\mathcal{T}_i^* \mathcal{T}_i + \lambda_i D^* D)^{-1} \mathcal{T}_i^* \mathcal{T}_i R^{i-1} S(t). \quad (4)$$

Thus, at each iteration, the operator-based approach must estimate the operator \mathcal{T}_i and the parameter λ_i in order to obtain the residual signals $R^i S(t)$.

The operator \mathcal{T}_i can be constructed by using a differentiation operator such as

$$\mathcal{T}_i = \sum_{k \in \mathbb{Z}} \alpha_i(k) \frac{d^k}{dt^k}, \quad (5)$$

where $\{\alpha_i(k)\}$, the parameters of the operator \mathcal{T}_i , is a square summable sequence belonging to $l_2(\mathbb{Z})$. In Ref. 8, a simpler form of the differential operator is proposed:

$$\left(\frac{1}{\varpi^2(t)} \frac{d^2}{dt^2} + 1 \right)^2, \quad (6)$$

where $\varpi(t)$ is the instantaneous frequency parameter at point t , which is obtained from *a priori* information about the signal or estimated from the signal itself. Because operator-based signal separation is an iterative process, the algorithm's parameters must be estimated correctly; otherwise, errors due to incorrect estimation of the current parameters will be propagated to sequent iterations. In this paper, our main objective is to derive a variational method that is capable of robust estimation of the instantaneous frequency of the operator.

3. Estimation of Instantaneous Frequency Parameters

We assume that the coherent component of a signal $S(t)$ is comprised of local narrow band signals that can be approximated as $\cos(\varpi t + \phi)$ in the neighborhood at any point t . Thus, a coherent component can be extracted by using the operator

$$\mathcal{T} = \frac{d^2}{dt^2} + \varpi^2(t), \quad (7)$$

where $\varpi(t)$ is the instantaneous frequency of the signal at t . To derive the operator, we need to estimate its instantaneous frequency parameters.

Let the signal $S(t)$ be decomposed into components $R(t)$ and $(S - R)(t)$, where $R(t)$ is called the residual signal. In addition, let $\alpha(t) = \varpi^2(t)$. Then, we search for $\hat{\alpha}(t)$ and $\hat{R}(t)$ such that

$$\{\hat{\alpha}(t), \hat{R}(t)\} = \arg \min_{\alpha, R} \left\{ \left\| \left(\frac{d^2}{dt^2} + \alpha(t) \right) (S - R)(t) \right\|^2 + \mu \|\mathcal{D}\alpha\|^2 \right\}, \quad (8)$$

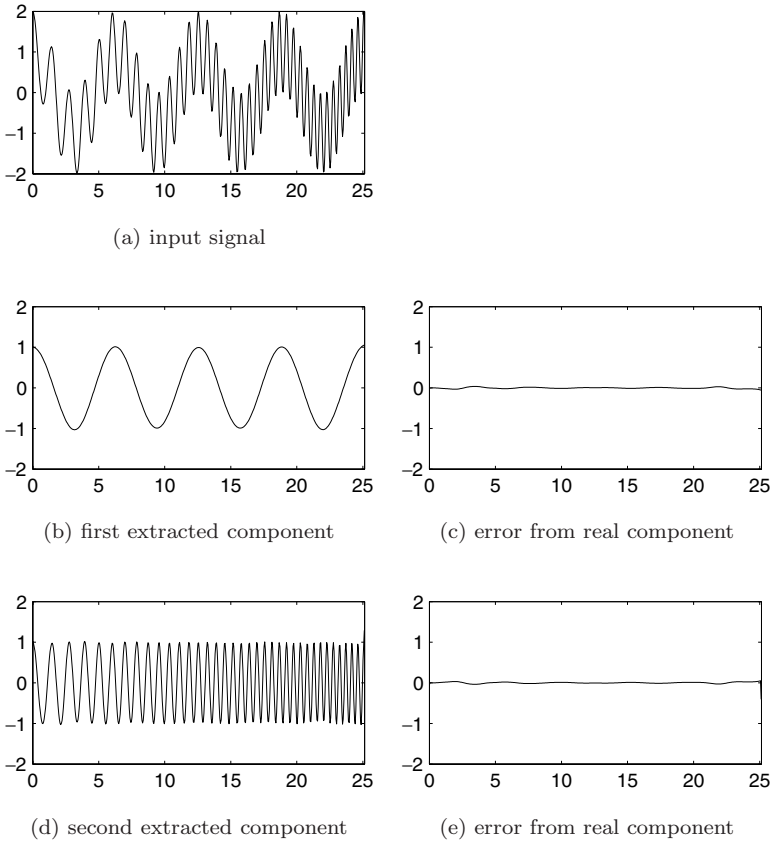


Fig. 1. Separation results of the proposed algorithm. (a) The signal is composed of a tone, $\cos(t)$, and a chirp signal, $\cos(4t + 0.2t^2)$. (b, d) the two extracted subcomponents, where (b) is the extracted tone signal and (d) is the chirp signal; (c and e) the error signals obtained by subtracting the extracted subcomponents from the corresponding correct subcomponents. The Lagrangian parameters used in the processing are $\lambda = 1e-5$ and $\mu = 5e-2$.

where \mathcal{D} is a regularized operator on α , and μ is the Lagrangian parameter. Note that \mathcal{D} is the second differential operator, which ensures that α is a smooth function.

In a discrete case, $S(t)$, $R(t)$ and $\alpha(t)$ can be represented as column vectors of length L , and \mathcal{D} can be represented as the matrix of the second difference. The optimization of Eq. (8) can be rewritten as

$$\{\hat{\alpha}, \hat{R}\} = \arg \min_{\alpha, R} \{ \|(D + P_\alpha)(S - R)\|^2 + \mu \|D\alpha\|^2 \}, \tag{9}$$

where P_α is a diagonal matrix in which the diagonal is equal to α . Let

$$\mathcal{F}(\alpha, R) = \|(D + P_\alpha)(S - R)\|^2 + \mu \|D\alpha\|^2 \tag{10}$$

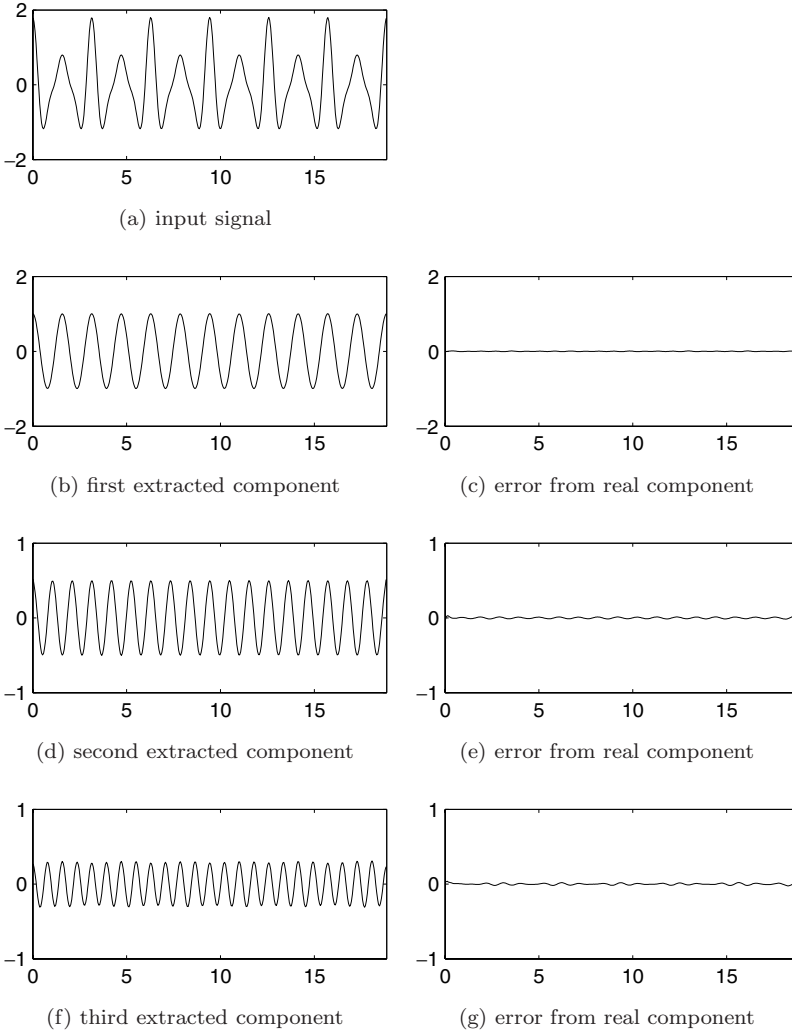


Fig. 2. The separation results derived by the proposed algorithm: (a) the input signal $S(t) = \cos(4t) + 0.5 * \cos(6t) + 0.3 * \cos(8t)$; (b, d and f) the three extracted components; (c, e, and g) the error signals obtained by subtracting the extracted subcomponents from the corresponding correct subcomponents, which are $\cos(4t)$, $0.5 * \cos(6t)$, and $0.3 * \cos(8t)$ for (b), (d), and (f), respectively. The parameter values used to obtain the first, second, and third components are, respectively, to be $\lambda = 5e-5$, $5e-5$, $3e-3$ and $\mu = 1e-2$, $1e-3$, $1e-3$.

then

$$\frac{\partial F}{\partial \alpha} = 2A'(A\alpha + D(S - R)) + 2\mu D'D\alpha,$$

where A is a diagonal matrix in which the diagonal is equal to $(S - R)$. Then, $\partial F / \partial \alpha = 0$ leads to

$$\hat{\alpha} = (A'A + \mu D'D)^{-1} A'D(S - R). \quad (11)$$

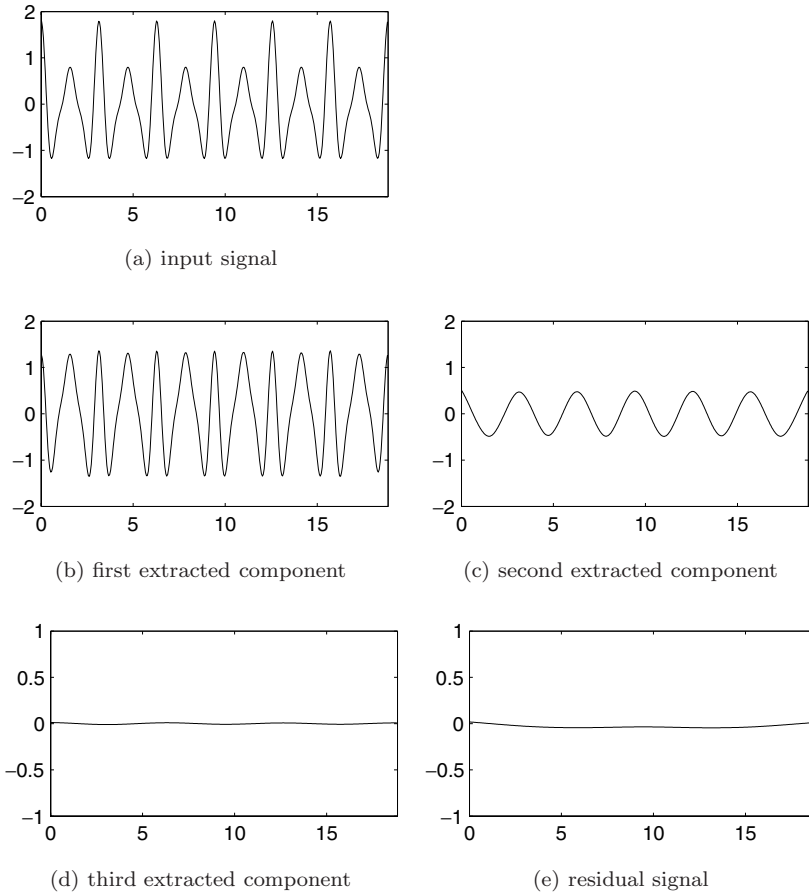


Fig. 3. The separation results obtained by using the EMD algorithm: (a) the input signal $S(t) = \cos(4t) + 0.5 * \cos(6t) + 0.3 * \cos(8t)$; (b–e) the components extracted by the EMD algorithm. The algorithm does not separate a signal into three additive subcomponents, as explained by analysis in.¹⁰

Similarly, $\partial F / \partial R = 0$ leads to

$$Q'Q(S - R) = 0, \tag{12}$$

where $Q = D + P_\alpha$. Equation (12) indicates that $S - R$ is in the null space of $Q'Q$. If the rank of $Q'Q$ is not full, then there are many solutions that would satisfy the equation. Here, we choose the solution adopted in Ref. 8

$$\hat{R} = (Q'Q + \lambda D^* D)^{-1} Q'Q S, \tag{13}$$

where λ is another Lagrangian parameter than μ . Thus, we can estimate the instantaneous frequency parameters and the residual signal by the following

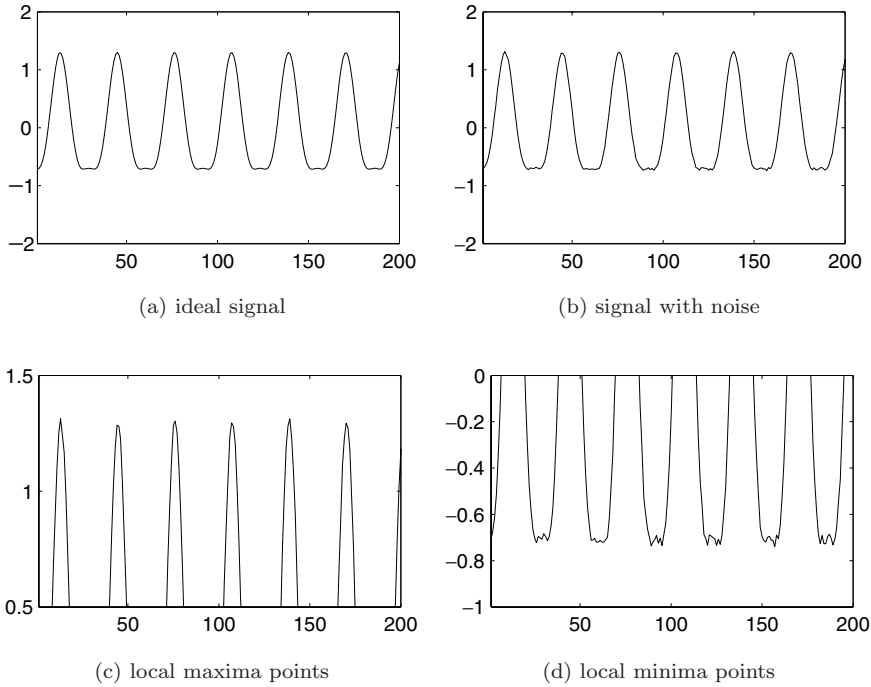


Fig. 4. (a) The signal is $\cos(4t) + 0.3 * \cos(8t)$; (b) the signal in (a) after adding a small amount of Gaussian noise; (c) the local maxima points of the signal in (b); (d) the local minima points of the signal in (b). In (d), several false local minima are created by noise.

alternative algorithm:

Algorithm

Step 1. Given S and D , manually choose μ and λ , initialize $j = 0$, $R_j = 0$;

Step 2. Estimate α_j by solving

$$\alpha_j = (A_j' A_j + \mu D' D)^{-1} A_j' D (S - R_j), \quad (14)$$

where A_j is a diagonal matrix in which the diagonal is equal to $(S - R_j)$.

Step 3. Estimating R_{j+1} by solving

$$R_{j+1} = (Q_j' Q_j + \lambda D' D)^{-1} Q_j' Q_j S, \quad (15)$$

where $Q_j = D + P_{\alpha_j}$, P_{α_j} is a diagonal matrix in which the diagonal is equal to α_j .

Step 4. Set $j = j + 1$ and repeat **Step 2** until $\|R_j - R_{j-1}\|$ is smaller than a given threshold.

Steps 2 and **3** estimate the operator's instantaneous frequency parameters and the residual signal alternatively. When the stop criterion is satisfied, $S - R_j$ is taken as the desired component. In the algorithm, the parameter λ is extremely important

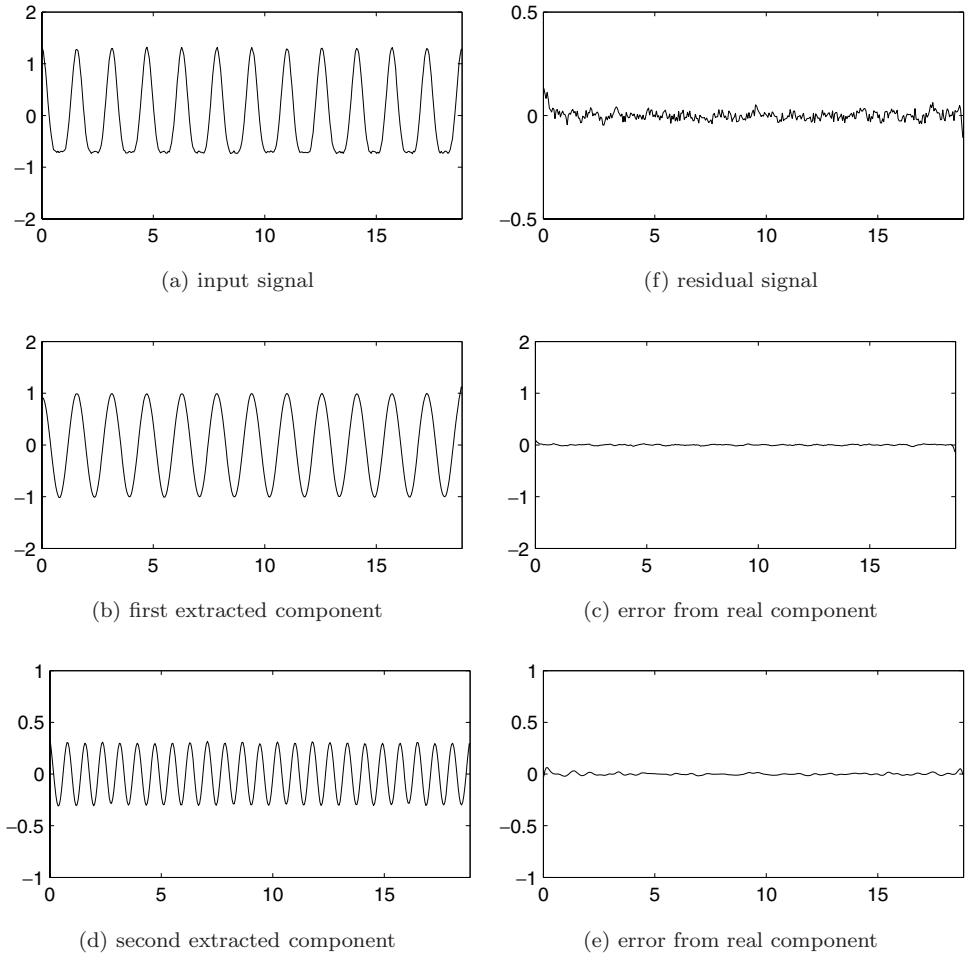


Fig. 5. The signal separation results derived by the proposed algorithm; (a) the input signal; (b and d) the two extracted subcomponents; (c and e) the error signals obtained by subtracting the extracted components from the correct corresponding subcomponents, which are $\cos(4t)$ in (b) and $0.3 * \cos(8t)$ in (d); and (f) the final residual signal. For the extraction of the first subcomponent, the values of Lagrangian multipliers of λ and μ are $5e-4$ and $1e-3$, respectively; and the second subcomponent, they are $1e-4$ and $1e-3$, respectively.

for correct signal separation. Until recently, its value had to be derived by trial and error, which was one of the drawbacks of operator-based signal separation algorithm. However, the problem was resolved by the analysis reported in Ref. 9. The other Lagrangian parameter μ is a smoothing regularization of α .

4. Experiment Results

In this section, we describe several signal separation experiments that apply different algorithms on synthetic and real-life data. In our experiments, the values of

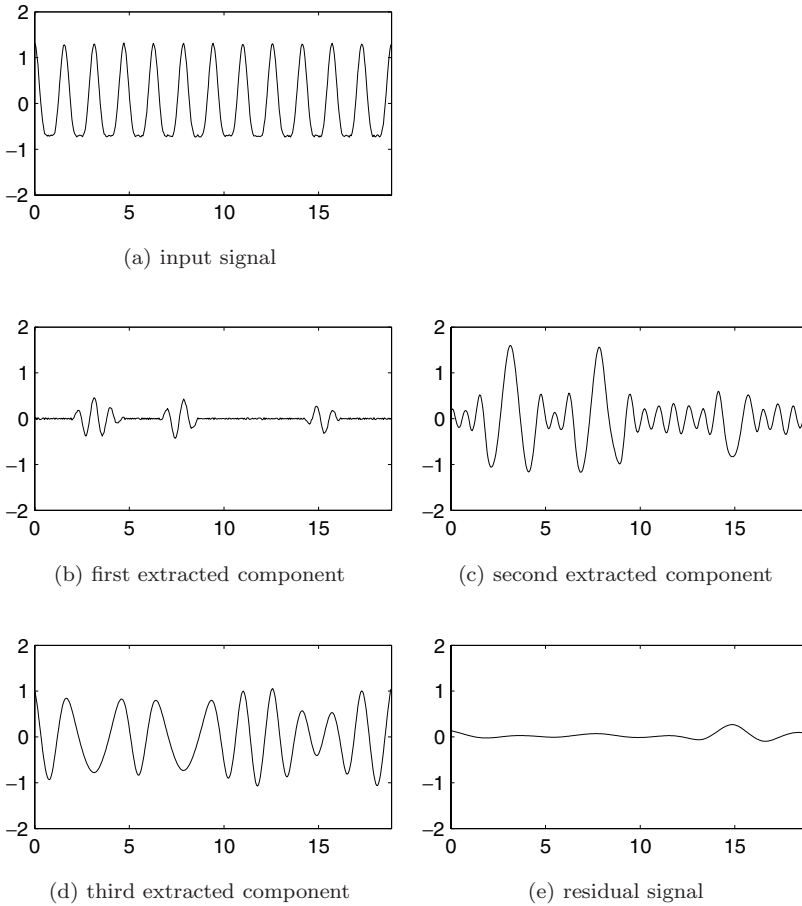


Fig. 6. The signal separation results derived by the EMD algorithm: (a) the input signal; (b, c, d) the three extracted subcomponents; (e) the final residual signal. The effects of a small amount of noise on the EMD decomposition results can be clearly observed in (b), (c), and (d). The algorithm fails to separate the noise from the clean signal.

parameters λ and μ are chosen manually. The suitable values of the parameters depend on the signal to noise level and the estimated instantaneous frequency.⁹ In the case of noiseless signals, both parameters are quite robust. However, when a signal is contaminated with a noise, the value of λ becomes more sensitive than that of μ . In our experiments, the values of λ were chosen between $1e-2$ and $1e-5$, and those of μ were between $1e-2$ and $1e-4$. In the following experiments, we use Neumann Boundary Conditions to symmetrically extend the boundaries of our signals. We used the EMD codes provided in Ref. 2 for our comparisons.

In the first experiment, the signal $S(t) = \cos(t) + \cos(4t + 0.2 * t^2)$, which is the summation of a tone and a chirp signal. The decomposition results in Fig. 1 shows that the tone and the chirp signals are well separated by our algorithm.

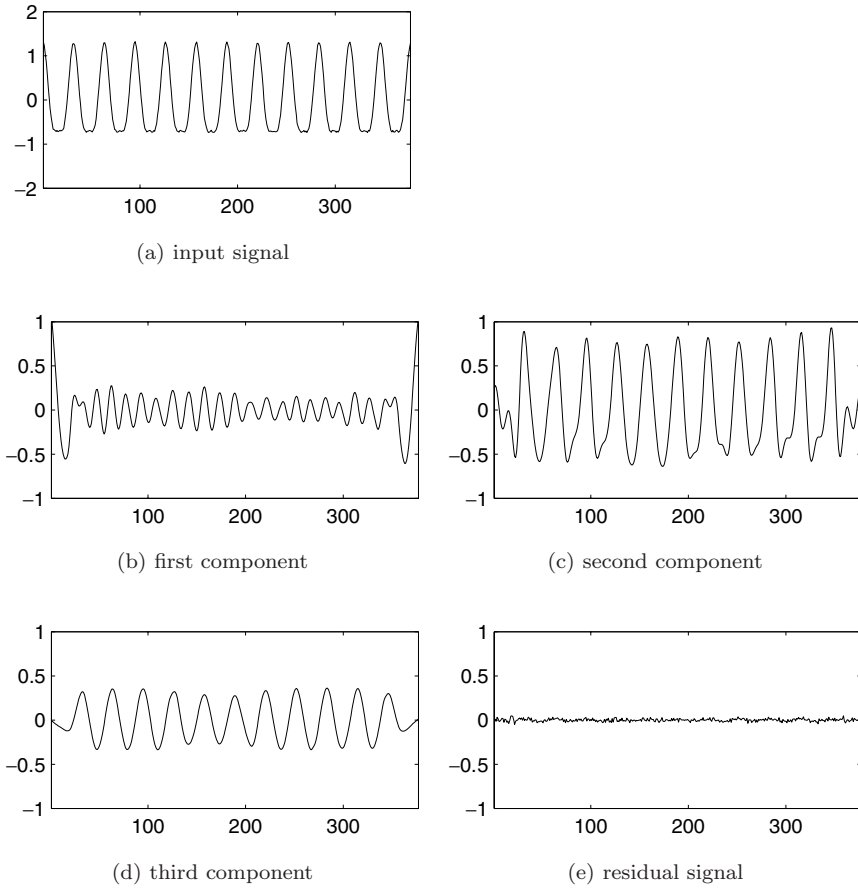


Fig. 7. The signal separation results obtained by using the second type of operator in Ref. 8: (a) the input signal. (b, c, d) the three extracted subcomponents; (e) the final residual signal. The decomposition result is not as good as that in Fig. 5 because the second type of operator uses the local extrema to derive the instantaneous frequency parameters.

In the second experiment, we compare the separation results of our algorithm with those derived by EMD for the harmonic signal $S(t) = \cos(4t) + 0.5 * \cos(6t) + 0.3 * \cos(8t)$. Note that in the EMD algorithm, the extrema are used to calculate the upper and lower magnitude envelopes of an oscillating signal. The subcomponents and the residual signal extracted by our algorithm are shown in Fig. 2. By selecting the correct values of μ and λ , our algorithm can separate the signal effectively. As described in Ref. 10, the values of a and f determine whether the EMD algorithm represents a signal of the following form, where $0 < f < 1$,

$$\cos(2\pi t) + a \cos(2\pi f t + \phi)$$

as the sum of two separate unmodulated tones or as a single modulated waveform. In this example, both $\cos(4t) + 0.5 * \cos(6t)$ and $\cos(6t) + 0.3 * \cos(8t)$ have a

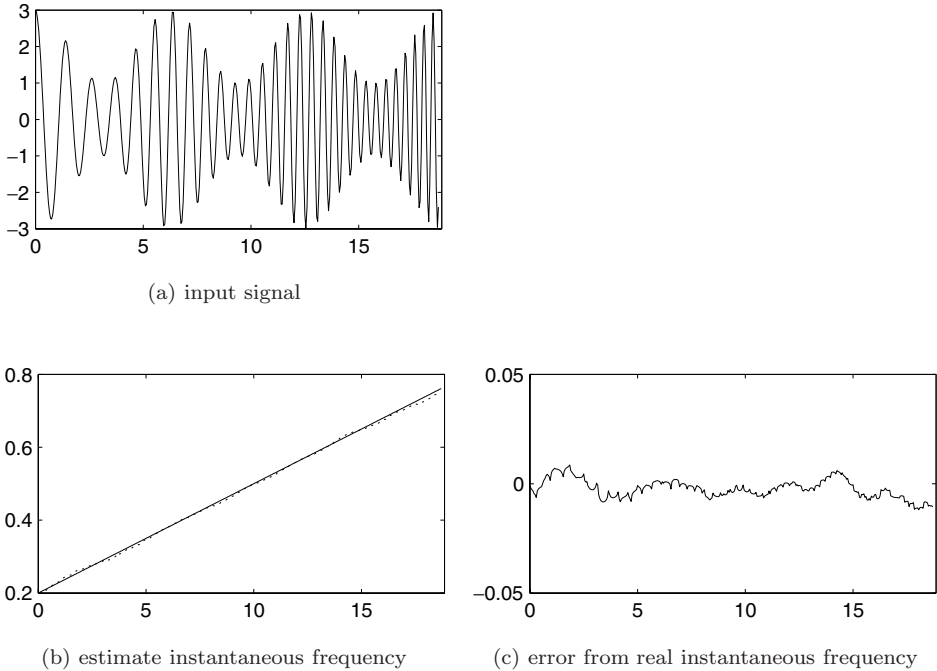


Fig. 8. Estimate the instantaneous frequency of an AM-FM signal using our proposed approach: (a) the input noise free signal. (b) the estimated instantaneous frequency and the real instantaneous frequency are plotted in dashed line and solid line, respectively. Since the two lines are very close to each other, the difference of them is shown at (c). The value of parameters λ and μ are $1e-3$ and $1e-3$, respectively. Both the estimated and the real instantaneous frequencies are multiplied by the sampling interval, which is $2\pi/126$ points.

frequency ratio f of larger than 0.67, which indicates that the EMD algorithm cannot separate any of the signal any further. Instead, as shown in Fig. 3, the EMD algorithm separated the signal into two IMF sub-components: $\cos(4t) + 0.3 * \cos(8t) + 0.5 * (\cos(6t) - \cos(2t))$ and $0.5 * \cos(2t)$.

The third experiment is designed to separate a noisy signal, obtained by adding a small amount of Gaussian random noise to the signal $S(t) = \cos(4t) + 0.3 * \cos(8t)$. As shown in Fig. 4, the locations of the local maxima are correct, but the locations of the local minima are strongly affected by the noise, which creates several false local minima. The false local extremal points make it difficult for both the EMD algorithm and the operator-based algorithm in Ref. 8 to estimate the correct envelopes and operators, respectively. The signal separation results of our algorithm, the EMD, and the algorithm proposed in Ref. 8 are shown in Figs. 5, 6, and 7, respectively. Note that the EMD method can handle the small additive noise after the appropriate pre-processing step.¹¹ We use the fourth example to illustrate the performance of our algorithm in estimating the instantaneous frequency of an AM-FM signal. In this example, the signal $S(t) = (2 + \cos(t)) \cos(4t + 0.3t^2)$, as

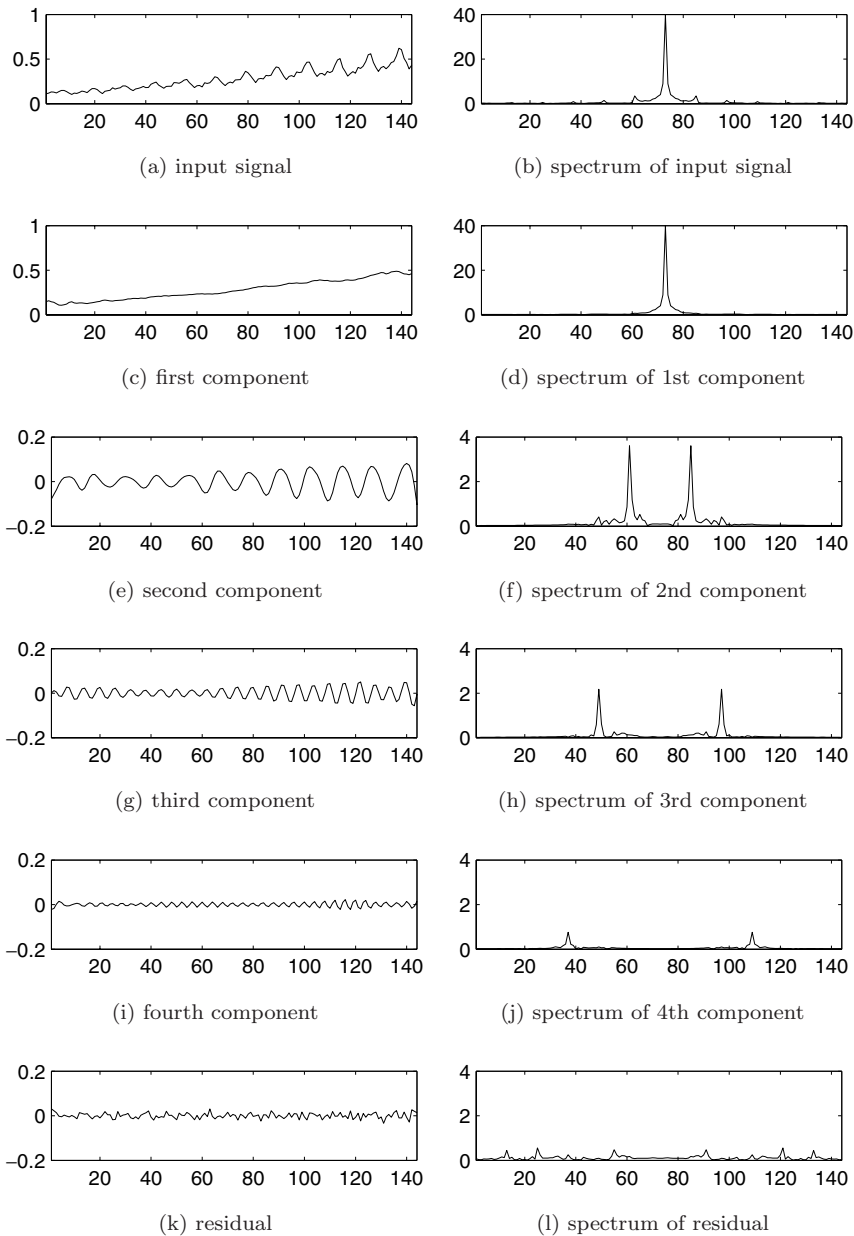


Fig. 9. Airline passenger data for a 12-year period (144 months). The proposed algorithm decomposes the data into four subcomponents and a residual: (a) the input signal; (b) the spectrum of the input signal; (c, e, g, i) the four extracted subcomponents; (d, f, h, j) the spectra of the four extracted components, (d), (f), (h) and (j) correspond to (c), (e), (g), and (i), respectively; (k, l) the last residual signal and its spectrum. To extract the first, second, third, and fourth subcomponents, the values of the Lagrangian parameter λ are $5e-3$, $1e-1$, $1e-1$ and $5e-2$, respectively; and the values of μ are $1e-3$, $5e-5$, $1e-4$, and $1e-3$ respectively. The first subcomponent indicates that the number of airline passengers increased progressively over the 12-year period. The peaks in the second subcomponents represent the peak season for air travel each year. The third and the fourth subcomponents are the variations of traveling passengers over the period.

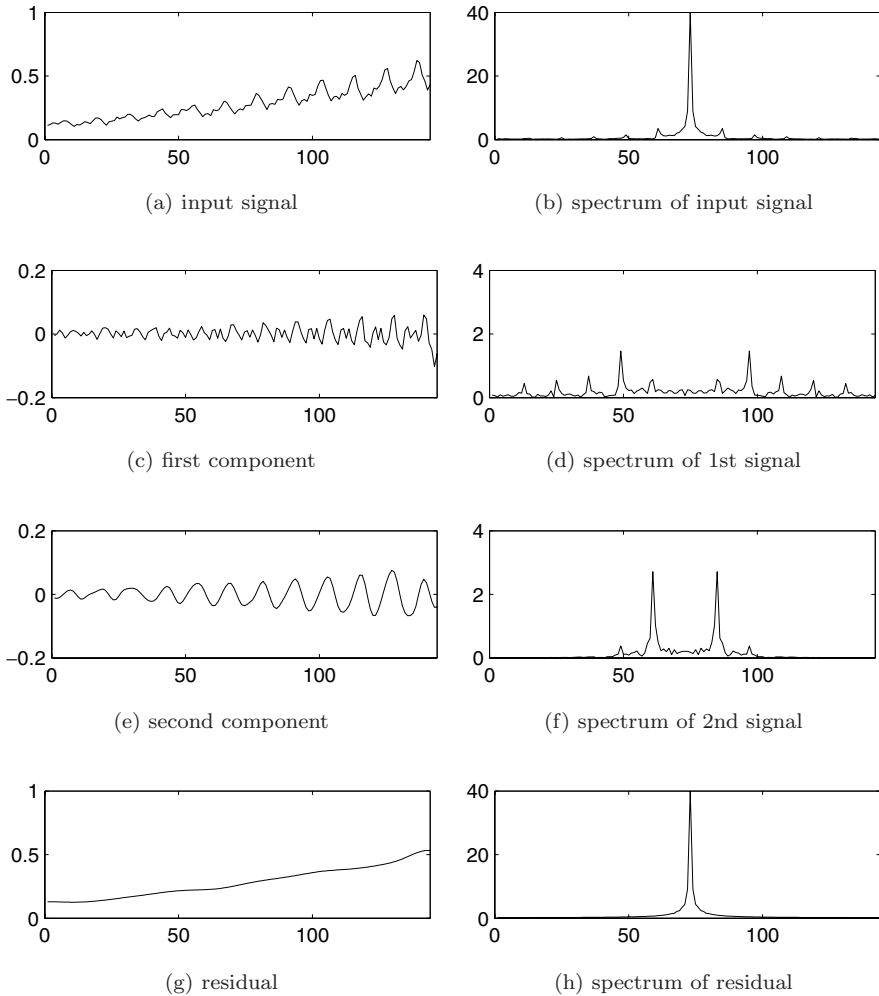


Fig. 10. Analyzing the airline passenger data with the EMD algorithm: (a) the input signal; (b) the spectrum of the input signal; (c, e) the two extracted subcomponents. (d, f) The spectra of the corresponding extracted subcomponents, where (d) and (f) correspond to (c) and (e) respectively; (g) the final residual signal; and (h) the spectrum of the final residual signal. The first extracted component is the variation of the passenger traveling. The peaks in the second component represent the peak periods for air travel each year. The final residual signal is the trend, which indicates that the number of airline passengers increased progressively over the 12-year period.

shown in Fig. 8(a). The estimated and the real instantaneous frequencies of the signal can be found in Fig. 8(b).

Finally, we compare the decomposition results of the EMD algorithm and our algorithm on airline passenger data obtained from Ref. 1. Figure 9 shows the decomposition results of deriving by applying our algorithm on the data. The first extracted subcomponent is the trend, and the other three subcomponents are the

narrow band frequency signals. From the first component, we can conclude that the number of passengers traveling by air increased during the 12 years covered by the data. In the second component, there are 12 peaks, indicating that there is one peak period for air travel each year. The third and fourth subcomponents are the variations in the number of passengers traveling. Figure 10 shows the decomposition results obtained by applying the EMD algorithm to the same data. There are two extracted subcomponents. The first is the variation of the traveling passengers. The second subcomponent has 12 peaks, each of which indicates the peak travel period in the respective year. The final residual signal is the trend of the signal, which indicates that the number of air passengers increased progressively over the 12 years covered by the data.

5. Conclusion

We have proposed a variational approach for estimating an operator's instantaneous frequency parameters in an operator-based signal separation method. The approach allows us to derive more robust signal separation results than those obtained by using the local extrema of a signal to estimate the operator's instantaneous frequency parameters. We also compare the decomposition results of the proposed algorithm with those of the EMD algorithm.

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