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PROPERTIES OF A NONLINEAR BLASCHKE PRODUCT DECOMPOSITION ALGORITHM

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Motivated by developments in nonlinear time–space–frequency analysis such as Refs. 8 and 14, we investigate the properties of Blaschke products. Inner products are constructed under which certain sets of Blaschke products, each have a single zero location, form orthonormal bases for $H^2(D)$. Using these sets of Blaschke products as approximants, a greedy algorithm decomposition is implemented. Properties are observed which may help to develop a faster search type algorithm.

Keywords: Analytic signal; instantaneous frequency; Blaschke product; time-frequency analysis.

1. Introduction

Blaschke products play an important role in the analysis of Hardy spaces on the disc and the upper half plane.[3,5] Recently, they have enjoyed new interest as a set of atoms for signal processing.[1,4,12–14] Much of this recent work has been motivated by the Hilbert–Huang Transform, developed by Huang *et al.*[8] This recent impetus to explore how Blaschke products may be applied to approximation has led to new research directions for the application of Blaschke products to time–scale–frequency analysis. This paper explores one method of using a special class of Blaschke products as atoms to perform decomposition of periodic signals.

The Hilbert–Huang Transform is a signal analysis tool which decomposes an input signal into component signals called intrinsic mode functions (IMFs) via a sifting algorithm. IMFs are implicitly defined by the stopping criteria of the sifting algorithm. These criteria essentially define an IMF to be a function which oscillates about 0 (the zero-crossing criterion), and has relatively slow amplitude modulation as well as local mean equal to 0 (the symmetry of the "envelopes"). With these criteria, an IMF attempts to capture the meaning of a "mono-component" signal. (This is a term that is used in signal processing literature, but seems to evade precise definition.) The IMFs are then used to construct analytic signals, which are then used to generate a spectrum of instantaneous frequencies. One question raised by the HHT is whether there is a class of functions having properties similar to IMFs which have a more natural mathematical structure. The set of analytic signals with nonnegative instantaneous frequency (ASNIFs)[13] seems to be such a family. The set of ASNIFs is a rather large class of functions, but it lends itself naturally to an incremental approach to building the set of atoms used: finite Blaschke products are an appropriate starting point. In recent papers[1,2,4,12–15] inner functions, specifically Blaschke products, have emerged as the beginnings of such a class of functions. They admit highly redundant representations of boundary value functions in $H^2(D)$ have positive instantaneous frequency. Even the set of all finite Blaschke products turns out to be too large than a set of a starting point, so we restrict ourselves in this paper to a smaller class of Blaschke products.

Consider the Hardy space on the unit disc $H^2(D)$. The Fourier basis modes for the boundary value functions are simply powers of z. Blaschke products are a natural extension of this set of atoms. The nonnegative powers of z are precisely those functions analytic on the unit disc D which have a finite number of zeros at the origin, and have an analytic extension to an open set $O \supset \{z\colon |z| \geq 1\}$. Finite Blaschke products are those functions which are analytic on the unit disc D, have a finite number of zeros in D, and have an analytic continuation to an open set $O \supset \{z\colon |z| \geq 1\}$. Thus Blaschke products are, in a sense, powers of z that have "let their zeros wander" in the unit disc D. This expansion of the set of Fourier atoms to Blaschke products has compelling motivations, especially for nonlinear approximation. The set of Blaschke products is much larger than the set of Fourier atoms, admitting the possibility of highly redundant representations. By investigating the properties of a rudimentary representation using certain Blaschke products as atoms, we hope to gain a better understanding of their potential for use in signal processing applications.

2. Definitions and Notations

D denotes the open unit disc in the complex plane $D = \{z\colon |z| < 1\}$.

The *Blaschke factor* for the unit disc with zero location a is the function

$$\zeta_a(z) = \frac{z-a}{1-\overline{a}z}.$$

The inverse function $\zeta_a^{-1}(z)$ will be referred to as an *inverse Blaschke factor*. $\zeta_a^{-1}(z)$ has the following property:

$$\zeta_a^{-1}(z) = \zeta_{-a}(z).$$

A *finite Blaschke product* for the unit disc is a function of the form $B(z) = \prod_{n=1}^{N} \zeta_{a_n}^{p_n}$, where $\{a_n\} \subset D$, and $p_1, p_2, \ldots, \in \mathbb{N}$.

$H^2(D)$ denotes the Hardy space of functions f, which are holomorphic on D and for which the following holds.

$$\sup_{0<r<1} \frac{1}{2\pi} \int |f(re^{i\theta}|^2 d\theta < 0.$$

If a function $f \in H^2(D)$ has a first derivative defined on the closed unit disc such that $\lim_{r\to 1} f'(re^{i\theta}) = -ie^{-i\theta}(d/d\theta)f(e^{i\theta})$ almost everwhere, then the *instantaneous frequency* $\omega_f(\theta)$ of f is defined on the boundary to be

$$\omega_f(\theta) = \Im\left(\frac{d}{d\theta}\log f(e^{i\theta})\right).$$

The boundary values of derivatives of elements of a Hardy space involve analysis which is outside of the scope of this paper.[7,9] We confine ourselves to a class of functions for which the existence of derivatives on the boundary is obviously not an issue: those functions holomorphic on D which have an analytic continuation to an open set $O \supset \{z : |z| \geq 1\}$. Instantaneous frequency shares some useful properties with logarithms. Specifically, the instantaneous frequency of a product is equal to the sum of the individual instantaneous frequencies. Thus, the instantaneous frequency of a Blaschke product is a sum of Poisson kernels.[1,2,4,12,13,15] For a Blaschke product $B(z) = \prod_{n=1}^{N} \zeta_{a_n}^{p_n}$ with zero locations $a_n = r_n e^{i\phi_n}$, the instantaneous frequency is equal to

$$\omega_B(\theta) = \sum_{n=1}^{N} p_n P_{a_n}(\theta), \quad P_{a_n}(\theta) = \frac{1 - r_n^2}{r_n^2 - 2r_n\cos(\phi_n - \theta) + 1}. \tag{1}$$

3. Blaschke Polynomials

A linear combination of Blaschke products $S = \sum_{j=1}^{k} c_j(\prod_{l=1}^{m_j} \zeta_{a_l}^{n_l})$ may be thought of as a polynomial in the variables $\bigcup_{j=1}^{k}\{\zeta_{a_1}, \ldots, \zeta_{a_{m_j}}\}$. Before doing so, one must verify that this identification is well-defined. That is, there is a one-to-one correspondence between each formal sum of the form $\sum_{j=1}^{k} c_j(\prod_{l=1}^{m_j} \zeta_{a_{j,l}}^{n_l})$ in many variables, and the corresponding complex function of one variable $f_S(z) = \sum_{j=1}^{k} c_j \prod_{l=1}^{m_j}(\zeta_{a_{j,l}}(z))^{n_l}$.

Definition 1. For each $p = \sum_{j=1}^{k} c_j(\prod_{l=1}^{m_j} \zeta_{a_{j,l}}^{n_l}) \in \mathbb{C}[\{\zeta_{a_{j,l}}; j = 1, \ldots, k; l = 1, \ldots, m_j\}]$, define $A_p = \{a_n : c_n \neq 0\}$ to be the set of all zero locations of the Blaschke factors in p.

Definition 2. Given a linear combination of Blaschke products

$$q(z) \in \operatorname{span}(\{\zeta_{a_{j,l}} : j = 1, \ldots, k; l = 1, \ldots, m_j\}),$$

define $P_q = \{z : q(z) \text{ has a pole at } z\}$.

Lemma 1. $|P_p| \leq \sum_{j=1}^{k} m_j$.

Proof. $p(z) = \sum_{j=1}^{k} c_j(\prod_{l=1}^{m_j} \zeta_{a_{j,l}}(z)^{n_l})$ is a linear combination of Blaschke products, each of which is holomorphic on $\mathbb{C}\backslash A_p$. Thus $|P_p| \leq |\{1/\overline{a_k}\}| = |A_p| = \sum_{j=1}^{k} m_j$. □

Lemma 2. $|P_p| = \sum_{j=1}^{k} m_j$

Proof. Let P be the set of all polynomials $p(\zeta_1, \ldots, \zeta_\mu) \in \mathbb{C}[\{\zeta_{a_{j,l}}, j = 1, \ldots, k, l = 1, \ldots, j\}]$ with nonzero degree for which $|P_p| < \sum_{j=1}^{k} m_j$. Choose $p \in P$ of minimal nonzero degree, and with a minimal number of Blaschke zero locations (i.e. $|A_p|$ is minimal). WLOG assume $|a_k| > 0,\ k = 1, 2, \ldots, \mu$. Choose $k \in \{1, 2, \ldots, \mu\}$ such that $(1/\overline{a_k}) \notin P_p$. Factor out as many powers as possible of ζ_k from those terms in p which are divisible by ζ_k.

$$p = \zeta_k^d p^* + q^*$$

$p(z)$ has no pole at $(1/\overline{a_k})$, and $q^*(z)$ is holomorphic at $(1/\overline{a_k})$, so $p^*(z)$ cannot have a pole at $(1/\overline{a_k})$. Now p^* is nonzero so q^* must be zero (or else p would not have a minimal number of terms in P). Thus $p = \zeta_k^d p^*$.

$p^*(z)$ cannot have a pole at $(1/\overline{a_k})$, or else $p(z)$ would have a pole at $(1/\overline{a_k})$ as well. However, $a_k \in A_{p^*}$, or else p would be a product of a function with a pole of degree d at $(1/\overline{a_k})$, and a function holomorphic at $(1/\overline{a_k})$; thus $p(z)$ would have a pole of degree d at $(1/\overline{a_k})$. Therefore $p^* \in P$, and p is not of minimal degree in P. □

Corollary 1. *The set of finite Blaschke products is linearly independent in* $H^p(D)$.

Now there is no problem using the term *Blaschke polynomial* to refer to a linear combination of finite Blaschke products, as each Blaschke polynomial is well-defined as a linear combination of Blaschke products.

4. Blaschke Factor Inner Products on $H^2(D)$

Definition 3. Given $a \in D$, the Blaschke factor inner product (BFIP) $\langle \cdot, \cdot \rangle_a$ on $H^2(D)$ is

$$\langle u, v \rangle_a = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} u(\zeta_a^{-1}(re^{i\theta}))\overline{v(\zeta_a^{-1}(re^{i\theta}))} d\theta. \tag{2}$$

When there is no subscript, the inner product is the usual inner product for $H^2(D)$. (Here the inner product is written with a radial limit. Typically in analysis on Hardy spaces such limits are understood to be non-tangential.[3])

$$\langle u, v \rangle = \langle u, v \rangle_0 = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\overline{v(re^{i\theta})} d\theta.$$

Whenever u and v have an analytic continuation to an open set $O \supset \{z : |z| \geq 1\}$, the radial limit can be dropped, giving

$$\begin{aligned} \langle u, v \rangle_a &= \frac{1}{2\pi} \int_0^{2\pi} u(\zeta_a^{-1}(e^{i\theta}))\overline{v(\zeta_a^{-1}(e^{i\theta}))} d\theta \\ &= \frac{1}{2\pi i} \int_{|z|=1} u(\zeta_a^{-1}(z))\overline{v(\zeta_a^{-1}(z))} \frac{dz}{z}. \end{aligned}$$

Equation (2) may be inconvenient to use for the evaluation of a Blaschke factor inner product. Some alternative forms are:

$$\begin{aligned}\langle u, v\rangle_a &= \lim_{r\to 1-}\frac{1}{2\pi i}\int_{|z|=r} u(\zeta_a^{-1}(z))\overline{v(\zeta_a^{-1}(z))}\frac{dz}{z}\\ &= \lim_{r\to 1-}\frac{1}{2\pi i}\int_{|z|=r} u(z)\overline{v(z)}\frac{\zeta_a'(z)}{\zeta_a(z)}dz\\ &= \lim_{r\to 1-}\frac{1}{2\pi i}\int_{|z|=r} u(z)\overline{v(z)}\frac{1-|a|^2}{(1-\overline{a}z)(z-a)}dz.\end{aligned}$$

Under sufficient regularity conditions, the radial limits may be dropped. For example, if $u(z)$ and $v(z)$ have analytic continuations to an open set $O \supset \{z\colon |z| \geq 1\}$, the integrals may be evaluated directly with $r = 1$, resulting in a simpler formulation:

$$\begin{aligned}\langle u, v\rangle_a &= \frac{1}{2\pi i}\int_{|z|=1} u(z)\overline{v(z)}\frac{1-|a|^2}{(1-\overline{a}z)(z-a)}dz\\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\overline{v(e^{i\theta})}\frac{1-|a|^2}{|e^{i\theta}-a|^2}d\theta.\end{aligned}$$

Definition 4. $H_a^2(D)$ denotes the set of complex valued functions holomorphic on D for which

$$\lim_{r\to 1-}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\left(\frac{1-|a|^2}{(1-\overline{a}re^{i\theta})(re^{i\theta}-a)}\right)re^{i\theta}d\theta\right)^{1/2} = \|f\|_a^2 < \infty.$$

It is well known that the mapping $T\colon H_a^2(D) \to H^2(D)$ defined by $T(u) = u \circ \zeta_a$ is an isometry between the two Hilbert spaces.[3,12,14] This mapping has some useful immediate consequences. $\langle\cdot,\cdot\rangle_a$ is clearly an inner product on $H_a^2(D)$, justifying the choice of terminology. Also it follows that $\{\zeta_a^n, n = 0, 1, \ldots\}$ is an orthonormal basis for $H_a^2(D)$ under the norm $\|u\|_a^2 = \langle u, u\rangle_a$, giving the following.

$$\langle\zeta_a^n, \zeta_a^m\rangle_a = \delta(n, m).$$

More generally, if $B(z) = \prod_{k=1}^K \zeta_{a_k}^{n_k}$, then we have (using the convention that $0^0 = 1$):

$$\langle B(z)^n, B(z)^m\rangle_a = \delta(n, m)B(a)^{|n-m|}.$$

Viewing $\cup_a H_a^2(D)$ as a union of isometric inner product spaces suggests an extension of the standard Fourier coefficients:

$$c_k^{a,b}(f) = \left\langle f, \zeta_a^k\right\rangle_b. \tag{3}$$

5. Powers of Single Blaschke Factors

Given the similarities between Blaschke products and the Fourier basis functions $\{1, z, z^2, \ldots\}$, it is natural to consider using Blaschke products as atoms for a decomposition algorithm. Using the zero locations of the atoms as a parameter space, the computational cost is immense compared to the fast Fourier transform (FFT). Performing an FFT using terms up to the nth power requires $n+1$ coefficients to be calculated at a cost of $O(n \log n)$. This gives the best nth degree polynomial approximation of an input signal. Now consider a Blaschke product decomposition where the zero locations of the Blaschke products used were restricted to some finite set of M points in D. Performing a greedy algorithm decomposition using all Blaschke products with zeros in the set M up to a power of n would require $\binom{M+n}{n}$ coefficients to be calculated at each iteration of the greedy algorithm. The cost of a brute force approach (calculating all possible coefficients) using arbitrary — or even a large set of — zero locations is impractically large: $\binom{M+n}{n}n$ complex multiplications. Even using the FFT-type acceleration described in Ref. 14 would only reduce this to $O(\binom{M+n}{n}\log n)$.

Two possible improvements to such an algorithm are an adaptive search technique instead of brute force minimization, and a further refinement of the-type algorithm in Ref. 14 which capitalizes on some further redundancy in the calculation. Either one of these improvements (or possibly both, if it makes sense) would help to bring such an algorithm closer to practical implementation. In this paper, we investigate the minimization functional in order to help determine what might be needed to perform an adaptive search. Such a search would be a very high dimensional search, and the functional used to construct the greedy algorithm may exhibit unexpected behavior that could interfere with a generic method such as conjugate gradients.

A practical starting point is to perform a decomposition into Blaschke powers for which each contain a single Blaschke factor. For this simplified case there is only one zero location to find at each iteration. At the mth step of the decomposition one must find a coefficient c, a zero location a, and an exponent p, which minimize the functional

$$J(c,a,p;f_m) = \frac{\|f_m - c\zeta_a^p\|_a^2 - \|f_m\|_a^2}{\|f_m\|_a^2}.$$

(The denominator is optional, but this normalized form is convenient for experimental purposes.) Note that the zero location a in the BFIP above is not constant, but rather depends on the Blaschke factor $\zeta_a(z)$ in question. Given a signal f, a Blaschke factor $\zeta_a(z)$, and an exponent p, the coefficient c which minimizes the J-functional can be found directly. The optimal coefficient $c^*(a,p;f)$ is the generalized Fourier coefficient $c^*(a,p;f) = c_p^{a,a}(f) = \langle f, \zeta_a^p\rangle_a$. Evaluating J at the optimal c value gives

$$\begin{aligned} J^*(a,p;f_m) &= \frac{|c^*|^2 - 2\ \mathrm{Re}(\langle c^*\zeta_a^p, f_m\rangle_a)}{\|f_m\|_a^2}, \\ &= \frac{|c^*|^2 - 2\ \mathrm{Re}(c^*\,\langle \zeta_a^p, f_m\rangle_a)}{\|f_m\|_a^2}, \end{aligned}$$

$$= \frac{|\langle f_m, \zeta_a^p \rangle_a|^2 - 2|\langle f_m, \zeta_a^p \rangle_a|^2}{\|f_m\|_a^2},$$

$$= -\frac{|\langle f_m, \zeta_a^p \rangle_a|^2}{\|f_m\|_a^2}.$$

6. Implementation

Given an input signal f, we use a BFIP decomposition which employs a greedy algorithm iteration:

$$\begin{aligned}
&\text{input } f \\
&g(0) \leftarrow f \\
&\text{for } m = 1{:}\,M \\
&A(m) \leftarrow \arg\min_{|a|<1} J(a, m; g(m-1)) \\
&C(m) \leftarrow c(A(m), n; g(m-1)) \\
&g(m) \leftarrow g(m-1) - C(m) \cdot \zeta_{A(m)}^m \\
&\text{endfor}
\end{aligned}$$

The outputs A and C store the location of the optimizer a, and the optimal coefficient value $c_n^{a,a}(f)$, respectively, for each iteration. The mth column of the array g stores the mth residual. The kth approximant f_k can be reconstructed from the coefficients.

$$f_k = \sum_{m=1}^{k} C(m) \cdot \zeta_{A(m)}^m.$$

The stopping criterion used for this investigation was $M = 10$. After the "good" components were extracted, the subsequent iterations exhibited quite rapid divergence of the residual norms. For experimental purposes, the behavior of the algorithm in the divergent regime was also of interest. In practice, some other stopping criterion may be preferable, such as one based on the residual norm — i.e. terminate if the norm of the residual does not decrease.

7. Examples

7.1. *A calibration exercise:* $f = \zeta_{1/2}$

Figure 1 shows a plot of the surface of the J-functional for $n = 1$. The minimizer is exactly $z = 1/2$ as desired, so the BFIP decomposition algorithm is exactly what one would expect: $f_1 = f$. It is not surprising that the algorithm recovers an exact decomposition for such a signal; that is hardly worth discussing. However, a simple observation about the J-functional's general behavior can be made from this case. The J-functional quite plainly approaches 0 almost everywhere on the boundary. It

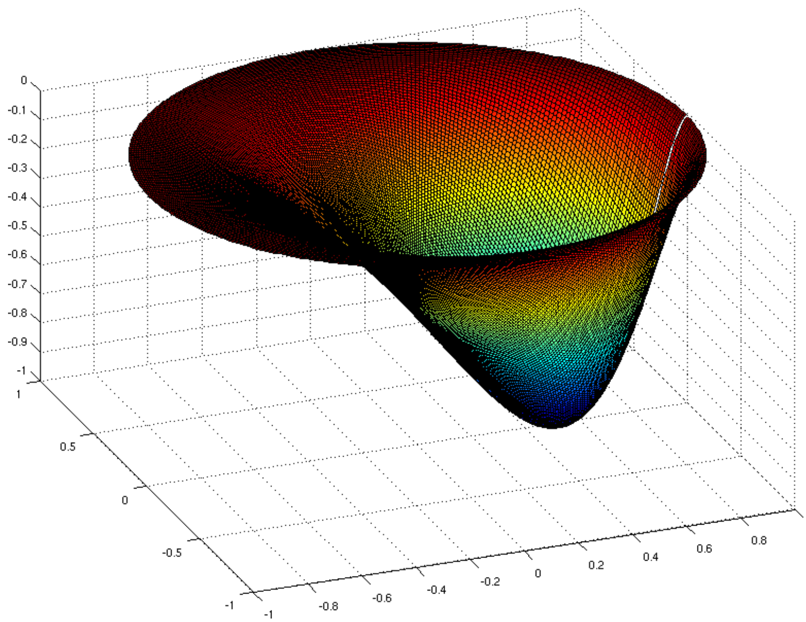

Fig. 1. $J^*(z, 1; \zeta_{1/2})$.

is trivial to prove that this happens for any input signal f whose values are given by a function which is analytic on an open set containing the unit disc D.

Lemma 3. *If $f(z)$ is analytic on an open set $O \supset \{z \colon |z| \leq 1\}$, then*

$$\lim_{r \to 1-} \langle f, 1 \rangle_{re^{i\phi}} = f(e^{i\phi}).$$

Proof. If $f(z)$ is analytic on an open set $O \supset \{z \colon |z| \leq 1\}$, then $|f|$ is uniformly bounded on the closed unit disc. Using the dominated convergence theorem and the continuity of f, the limit may then be brought inside f:

$$\begin{aligned}
\lim_{|r| \to 1-} \langle f, 1 \rangle_{re^{i\phi}} &= \lim_{|r| \to 1-} \frac{1}{2\pi} \int_0^{2\pi} f(\zeta_{re^{i\phi}}^{-1}(e^{i\theta})) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f\left(\lim_{r \to 1-} \zeta_{-re^{i\phi}}(e^{i\theta}) \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) d\theta \\
&= f(e^{i\phi}).
\end{aligned}$$

□

This immediately implies that $\lim_{r \to 1} J(re^{i\phi}, 0; f) = f(e^{i\phi})$. Furthermore, because the Blaschke powers $\{\zeta_a^n, n = 0, 1, 2, \ldots\}$ form an orthonormal basis under $\| \cdot \|_a$, all the other coefficients must approach zero.

$$\lim_{r \to 1} J(re^{i\phi}, n; f) = 0 \quad \text{for } n = 1, 2, \ldots \tag{4}$$

This guarantees the existence of a minimizer on D for each $n > 0$. For $n = 0$, this result actually ensures that a minimizer is guaranteed not to exist inside D. This is not detrimental because we are primarily interested in analyzing the oscillatory part of the signal. The $n = 0$ coefficient represents a constant term found by evaluating a weighted average of the function values around the disc. From the perspective of time–scale–frequency analysis, this is the least interesting coefficient of all. By restricting the search to terms with degree $n > 0$, a decomposition could be obtained which would have a minimizer at every step, and hopefully give a useful description of the frequency content of a signal.

7.2. *A nonunique minimizer:* $f = \zeta_{1/2} - \zeta_{-1/2}$

If the J-functional could be shown to have a unique minimizer (or better yet, to be convex on D), then it would be a relatively simple matter to implement a fast iterative algorithm to find the minimum very quickly. Unfortunately this is not the case, as can be clearly shown by decomposing the input signal $f = \zeta_{1/2}(z) - \zeta_{-1/2}(z)$. This signal also illustrates the failure of the greedy algorithm to recover a sparse representation of the simplest nontrivial linear case possible: a two term Blaschke polynomial. The two minimizers do not coincide with the points $z = \pm(1/2)$. Instead, they are at $z = \pm 0.7750$, as shown in Fig. 2.

At first glance, it would seem that any "good" Blaschke product decomposition technique ought to be able to recover the two term decomposition of f exactly.

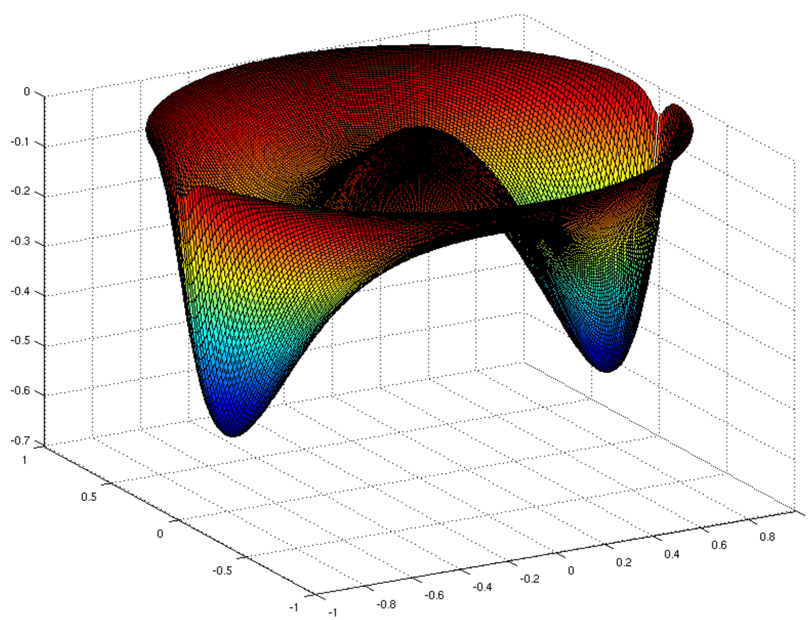

Fig. 2. $J^*(z, 1; \zeta_{1/2} - \zeta_{-1/2})$.

Thus it may be disappointing to find that the greedy algorithm not only fails to terminate at two terms, but also fails to identify the zero locations correctly. Closer investigation reveals that things are not too bad, and not that surprising.

In the case of the Fourier series, orthogonality guarantees that every convergent representation of a signal has a minimal number of terms — because there is only one convergent series. Every nonzero coefficient is necessary. Maximizing the rate of convergence thus simply requires that one choose the largest coefficients first. With a redundant set of atoms, this is not the case in general. In this example, an algebraically sparse (exact) representation exists using only two terms. However, the rate of convergence of that representation is not maximal at every step. The ratio of norm of the first residual to the norm of the input signal is $(\|r_1\|_0/\|f\|_0) = 0.625$ for the algebraically sparse representation, and $(\|r_1\|_0/\|f\|_0) = 0.416$ for the greedy algorithm. While not a huge improvement, it is significant. In retrospect, it would be unrealistic to expect a greedy algorithm to recover algebraic sparsity when it is specifically tailored to minimize the residual at each step, rather than minimize the number of terms in the approximation. If algebraic sparsity is desired, one possible solution is to find the best n-term approximant at the nth step, but that is a much more difficult task than simply finding one term at each step.

7.3. *Boundary sampling behavior: $f = z$*

Certain artifacts appear near the boundary of the unit circle due to sampling effects. This is not surprising, as similar artifacts are produced by most signal processing techniques. However, if a search algorithm is ever to be developed for a Blaschke product decomposition, it is vital that a "safe" search region be determined which is devoid of artifacts arising from computational errors. The computed functional exhibits wild oscillations near the boundary of D which may produce minima that do not exist in the true functional, but are a result of sampling phenomena. For convenience in the preliminary investigation of such effects, the input signal $f(z) = z$ is used because it is simple to evaluate the exact value of the J-functional.

$$
\begin{aligned}
J(a, n; z) &= -\frac{|\langle \zeta_a^n, z\rangle|^2}{\|z\|_a^2} \\
&= -|\langle \zeta_a^n, z\rangle|^2 \\
&= -\left| \frac{1}{2\pi i} \int_{\partial D} \left(\frac{z-a}{1-\overline{a}z} \right)^n \overline{z} \left(\frac{1-|a|^2}{(1-\overline{a}z)(z-a)} \right) dz \right|^2 \\
&= -\left| \frac{1}{2\pi i} \int_{\partial D} \left(\frac{z-a}{1-\overline{a}z} \right)^n \left(\frac{1-|a|^2}{(1-\overline{a}z)(z-a)} \right) \frac{dz}{z} \right|^2 \\
&= \begin{cases} -|a|^2, & n = 0 \\ -|a|^{2n-2}\left(1-|a|^2\right)^2, & n = 1, 2, 3, \ldots \end{cases}
\end{aligned}
$$

The entire J-functional was computed using $k = 90, 360, 1440$, and 5760 sample points for $f(z)$. In each case, the J-functional was evaluated using a mesh of values for a with very high resolution near the boundary. Plots of error as a function of distance from the boundary are shown in Fig. 3. Each plot shows the computed error for every point on the surface of the J-functional. That is, for each value $x =$ "distance from the boundary", the plot shows all the error values of the J-functional on the circle of radius $r = 1 - x$.

For all resolutions and exponents n, the plots have a similar profile. In each case the errors are within machine accuracy (in this case, below about 10^{-30}) in some region near the center. Near the boundary there is a pronounced ramp up to unacceptable levels where sampling effects become significant (subsequently called "the ramp"). In the case $K = 90, n = 1$, this occurs at approximately 0.36 from the boundary. After identifying the beginning of the ramp in error values near the boundary a clear pattern emerges: the location of the ramp appears to be inversely proportional to the number of sample points k. That is, increasing the sampling rate of the signal by a factor of C decreases the width of the "unsafe" region by approximately the same factor. This is useful to observe in order to estimate the resolution needed to establish a given "safe" search region. Of course, in order for such a region to be truly safe, a result would have to be proven rather than conjectured, but such a result is not available yet.

A similar pattern can be observed for the exponent $n = 10$. In this case there is a very pronounced ramp in error values, but the profile of the ramp itself is much less angular. However the same proportionality appears to be in effect as in the case $n = 1$. Each increase in resolution by a factor of 4 appears to decrease the distance from the boundary to the ramp in error values by a factor of approximately 4. It appears likely that a linear relationship exists between the distance between sample points and the width of the computationally "bad" region of relatively high error near the boundary. The relationship between the exponent n and the width of the "bad" region does not seem to be so simple. After observing the ramp in error profile for exponents other than 1 and 10, it is clear that the distance from the boundary increases with the exponent but the dependence is not linear. Further investigation is needed to form a useful hypothesis about this relationship.

8. Conclusions and Future Research

Properties of Blaschke products were observed as they relate to approximation theory. Linear independence of Blaschke polynomials was established justifying the term *Blaschke polynomial*. Blaschke factor inner products were introduced as a convenient terminology and notation for the approximation discussion to follow. Blaschke polynomials were then used to perform a greedy algorithm signal decomposition. From this decomposition, some observations were made:

- The J-functional approaches 0 near the boundary of D.
- The J-functional does not have a unique minimizer.

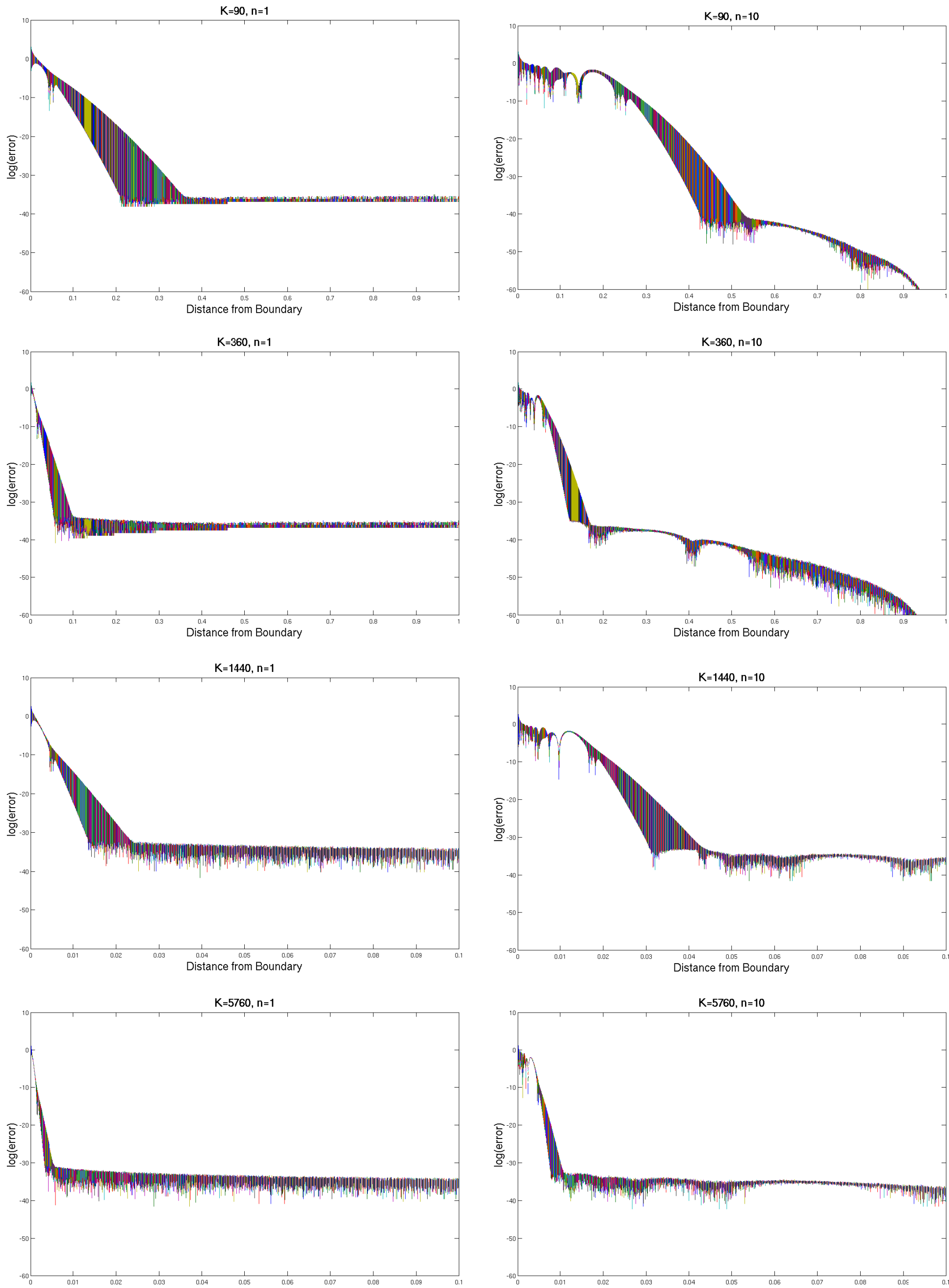

Fig. 3. Computational error profiles for $N = 90, 360, 1440$, and 5760 sample points at exponents $n = 1$ and $n = 10$.

- The J-functional exhibits sampling behavior near the boundary of D which must be taken into consideration when searching for its minimizer.

There are some classical results concerning the linear span of Blaschke products which relate to approximation. Blaschke products generate H^∞,[10] and more specifically H^∞ is the closed linear span of interpolating Blaschke products.[6] The nonexistence of a best approximant can be deduced from the fact that any bounded analytic function on D with Riemann integrable boundary values can be approximated with arbitrary precision by a linear combination of (possibly infinite) Blaschke products.[11] These results use infinite Blaschke products which may prove impractical for computational purposes. However, insights gained from how interpolating Blaschke products behave may help to inform the development of algorithms which search for "good" (in some sense) Blaschke polynomial approximants.

A decomposition into more general families of Blaschke products is a desirable goal for future development, but even a setting as simple as the set of powers of single Blaschke factors $\{\zeta_a^n; a \in D, n = 0, 1, \ldots\}$ has the potential for meaningful insights. From Eq. (1), the radius of a zero location determines the scale of the feature: the closer a zero location is to the boundary of the circle, the more concentrated the instantaneous frequency. The polar angle of a zero location determines its position on the boundary (i.e. time). The degree of the Blaschke product determines its wave-number, or frequency. Thus, even though $\mathcal{S}_1$ is a very small subclass of the set of finite Blaschke products for the disc, it has the potential to give a time–scale–frequency analysis of analytic signals. Generalizing the class of atoms further would yield an even richer analysis than this.

Because they are a natural extension of the classical Fourier basis functions, Blaschke products have many desirable properties for use in signal processing. However, Blaschke products present major theoretical and computational challenges. The set of atoms is very large — even much larger than the set of approximants used in this paper. It seems that a high dimensional search would be required in order to perform decomposition or approximation using any larger set of Blaschke products than was used here, or even to perform decomposition using $\mathcal{S}_1$ at a higher resolution. On the theoretical side, there are many results which remain undiscovered: what characterizes the best Blaschke polynomial approximants with m terms of degree at most n? Is there a useful characterization of signals based on the rate of convergence of the best Blaschke polynomial approximants? What does a complete sampling theory for Blaschke products look like? Are there fast algorithms available for performing a Blaschke polynomial decomposition: either FFT-type as in Ref. 14, or using adaptive search, or both?

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