Advances in Adaptive Data Analysis
Vol. 1, No. 3 (2009) 373–405

FAST NONLINEAR FOURIER EXPANSIONS

RUI WANG
School of Information Science and Engineering
Graduate University of the Chinese Academy of Sciences
Beijing 100190, P. R. China
wangrui@gucas.ac.cn

YUESHENG XU[*,†,‡] and HAIZHANG ZHANG[*,§]
[*]*Department of Mathematics, Syracuse University*
Syracuse, NY 13244, USA
and
[†]*Academy of Mathematics and Systems Science*
Chinese Academy of Sciences, Beijing 100190, P. R. China
[‡]*yxu06@syr.edu*
[§]*hzhang12@syr.edu*

Motivated by the analytic signal approach and general construction methods by Qian *et al.* (accepted by *Adv. Comput. Math.*), we construct a class of orthonormal bases for the real signal space $L^2_{\mathbb{r}}[0, 2\pi]$, which have nonconstant physically meaningful instantaneous frequencies. We develop a fast algorithm for computing the Hilbert–Fourier expansion of a given function in terms of the orthonormal bases. Moreover, we study the approximation properties of the Hilbert–Fourier expansion. A numerical example is presented to demonstrate an adaptive Fourier expansion based on an optimal selection of the parameter a in the orthonormal bases according to the approximation error.

Keywords: Instantaneous frequency; analytic signal; orthonormal bases; Hilbert–Fourier bases; fast nonlinear Fourier transform.

1. Introduction

The classical Fourier analysis is a linear process. The traditional Fourier basis functions have constant frequencies. It has limitations for processing a nonlinear and nonstationary signal. Recently, there was a great effort[1–6] to address this issue. In order to overcome the shortcoming of the Fourier analysis, many methods have been proposed. Among them, the analytic signal approach in Ref. 2 was used to investigate the instantaneous amplitude, phase, and frequency. Through the analytic signal approach, the empirical mode decomposition (EMD) algorithm was developed in Ref. 3 to provide a numerical algorithm to adaptively decompose a

[‡]Corresponding author.

signal into a finite sum of intrinsic mode functions (IMFs) each of which has physically meaningful instantaneous frequency. As more and more evidences show that the EMD algorithm works very well, understanding its mathematical insight and building its mathematical foundation become pressing.

There are two issues in building the mathematical foundation for the EMD algorithm. The first one is to construct a large amount of functions with physically meaningful instantaneous frequency. The second one is to establish an adaptive and fast algorithm to decompose a signal into a sum of functions constructed in the last stage. Some efforts for the first issue have been made in Ref. 7. In particular, general constructions of orthonormal bases with nonnegative and nonconstant instantaneous frequency for the Hilbert space of real square integrable functions were established. Based on the work of Ref. 7, this paper aims at the second issue. Specially, following the general construction described in Ref. 7, we construct a class of orthonormal bases which we call the A-system for the real signal space $L_r^2[0, 2\pi]$, which have nonconstant physically meaningful instantaneous frequencies. To develop a fast algorithm for computing the *Hilbert–Fourier expansion* of a given function in terms of the orthonormal bases, we introduce the notion of A-conjugate pair of a given function and establish a relationship between the Hilbert–Fourier coefficients and the classical Fourier coefficients. We also investigate the approximation properties of the Hilbert–Fourier expansion.

We organize this paper in eight sections. In Sec. 2, we construct a class of orthonormal bases with nonnegative and nonconstant instantaneous frequency for $L_r^2[0, 2\pi]$. Section 3 is devoted to a study of properties of the zeros and the extrema of the basis functions. Specifically, it will be shown that the basis functions satisfy the first condition of an IMF.[3] In Sec. 4, we introduce the notion of the A-conjugate pair of a given function in $L_r^2[0, 2\pi]$. We introduce in Sec. 5 the nonlinear Fourier series by using the A-system and investigate various properties of the series. In Sec. 6, we establish the approximation order of the Hilbert–Fourier expansion and in Sec. 7, we develop a fast algorithm for computing the Hilbert–Fourier expansion. A numerical example is presented in Sec. 8 to demonstrate an adaptive Fourier expansion method based on an optimal selection of the parameter a in the A-system according to the approximation error. The numerical results show that by choosing the optimal parameter a adaptive to a given function, we can improve significantly the approximation accuracy.

2. Orthonormal Bases for $L_r^2[0, 2\pi]$

In this section, we describe a construction of a class of orthonormal bases for $L_r^2[0, 2\pi]$, each of which has nonconstant physically meaningful instantaneous frequency. These bases are constructed by using the analytic signal approach.

We first recall the analytic signal approach. The Hilbert transform plays an important role in this method. In the case of time-frequency analysis of periodic but nonlinear signals, the circular Hilbert transform $\tilde{H}$ is defined for each $f \in L^1[0, 2\pi]$

at $t \in [0, 2\pi]$ as

$$(\tilde{H}f)(t) := \text{p.v.}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\cot\frac{s}{2}ds := \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\varepsilon\le|s|\le\pi} f(t-s)\cot\frac{s}{2}ds, \tag{2.1}$$

whenever the Cauchy principal value of the above singular integral exists. Here $L^p[0, 2\pi]$, $1 \le p < \infty$, denotes the set of all the 2π-periodic functions such that $\int_0^{2\pi} |f(t)|^p dt < +\infty$. If $f \in L^2[0, 2\pi]$ then $\tilde{H}f$ has the following form in terms of the Fourier coefficients of f

$$(\tilde{H}f)(t) = \sum_{k\in\mathbb{Z}} -i\,\text{sgn}(k)c_k(f)e^{ikt}, \quad \text{a.e. } t \in [0, 2\pi], \tag{2.2}$$

where $\text{sgn}(k)$ takes value -1, 0, and 1, respectively for $k < 0$, $k = 0$, and $k > 0$, and $c_k(f)$ is the kth Fourier coefficient of f defined by

$$c_k(f) := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt. \tag{2.3}$$

Set $L_r^2[0, 2\pi] := \{f : f \in L^2[0, 2\pi], f \text{ is real}\}$. It is clear that $L_r^2[0, 2\pi]$ is a Hilbert space over $\mathbb{R}$. The analytic signal associated with a function $f \in L_r^2[0, 2\pi]$ is defined by

$$\tilde{A}f := f + i\tilde{H}f. \tag{2.4}$$

Let $\tilde{A}f$ be further written as

$$(\tilde{A}f)(t) = \rho(t)e^{i\theta(t)}, \quad t \in [0, 2\pi], \tag{2.5}$$

where $\rho \ge 0$ and θ are real functions. Hence f has the following amplitude–frequency modulation

$$f(t) = \rho(t)\cos\theta(t), \quad t \in [0, 2\pi].$$

Then, $\rho(t)$ and $\theta(t)$ are defined as the instantaneous amplitude and phase of the signal f at time t, respectively. The instantaneous frequency so defined is physically meaningful only if the derivative $\frac{d\theta}{dt}$ is nonnegative. A theoretical method of constructing functions with nonnegative instantaneous frequencies has been proposed in Ref. 8. It leads to solving functions $\rho \in L^2[0, 2\pi]$ and 2π-periodic real function $\theta \in C^1(\mathbb{R})$ from the nonlinear singular integral equation

$$\tilde{H}(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t), \quad \text{a.e. } t \in [0, 2\pi], \tag{2.6}$$

subjected to the constraint

$$\rho(t) \ge 0, \quad \frac{d\theta(t)}{dt} \ge 0, \quad t \in [0, 2\pi].$$

Solutions of this singular integral equation have been characterized in Ref. 9 as the boundary values of functions in Hardy spaces and star-like functions in complex analysis.

To continue our discussion, we need the notion of the Hardy space (cf., Refs. 10–12). Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{N}_n := \{1, 2, \ldots, n\}$. Set $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$. We denote by $\mathbf{H}(\mathbb{U})$ the set of the holomorphic functions defined on $\mathbb{U}$. The Hardy space $\mathbf{H}^2(\mathbb{U})$ is then defined by

$$\mathbf{H}^2(\mathbb{U}) := \left\{ f \in \mathbf{H}(\mathbb{U}) : \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty \right\}.$$

It is well known that each $f \in \mathbf{H}^2(\mathbb{U})$ has the representation

$$f(z) = \sum_{n\in\mathbb{Z}_+} a_n z^n,$$

where $\sum_{n\in\mathbb{Z}_+} |a_n|^2 < \infty$. The Hardy space is a Hilbert space endowed with the inner product defined in terms of the nontangential limit of functions in the space. To see this, we now recall the definition of the nontangential limit. For $0 < \alpha < 1$, the nontangential approach region with vertex e^{it}, $t \in [0, 2\pi]$, is defined by

$$\Omega_\alpha(e^{it}) := \{\lambda e^{it} + (1-\lambda)z : \lambda \in (0,1), |z| < \alpha\}.$$

If $f \in \mathbf{H}^2(\mathbb{U})$ then there exists a $f^* \in L^2(\partial\mathbb{U})$ such that for any $0 < \alpha < 1$ and almost every $t \in [0, 2\pi]$

$$\lim_{\Omega_\alpha(e^{it})\ni z\to e^{it}} f(z) = f^*(e^{it}).$$

We call f^* the nontangential limit of f. The inner product of the Hardy space $\mathbf{H}^2(\mathbb{U})$ is now defined for $f, g \in \mathbf{H}^2(\mathbb{U})$ by

$$\langle f, g\rangle_{\mathbf{H}^2(\mathbb{U})} := \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{it})\overline{g^*(e^{it})} dt.$$

It is known from Ref. 7 that an orthonormal basis for $L^2_r[0, 2\pi]$ having physically meaningful instantaneous frequencies can be derived from an orthonormal basis for the Hardy space $\mathbf{H}^2(\mathbb{U})$. Specifically, if $1, m_j, j \in \mathbb{N}$, with nontangential limits

$$m_j^*(e^{it}) = \rho_j(t)e^{i\theta_j(t)}, \quad t \in [0, 2\pi], \quad j \in \mathbb{N},$$

satisfying $\theta'_j \geq 0$, $j \in \mathbb{N}$, form an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$, then functions

$$\sqrt{\frac{1}{2\pi}}, \quad \sqrt{\frac{1}{\pi}}\rho_j \cos\theta_j, \quad \sqrt{\frac{1}{\pi}}\rho_j \sin\theta_j, \quad j \in \mathbb{N},$$

constitute an orthonormal basis for $L^2_r[0, 2\pi]$ having physically meaningful instantaneous frequencies. This leads us to a construction of orthonormal bases for $\mathbf{H}^2(\mathbb{U})$.

A general construction of orthonormal bases for $\mathbf{H}^2(\mathbb{U})$ based on outer functions was proposed in Ref. 7. We now recall the definition of an outer function in Hardy

Spaces. According to Ref. 12, a function f is called an outer function, if there exists a positive measurable function ϕ on $\partial\mathbb{U}$ with $\log\phi \in L^1(\partial\mathbb{U})$ such that

$$f(z) = c\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\phi(e^{it})dt\right\}, \quad z \in \mathbb{U},$$

where c is a unimodular constant. A function $f \in \mathbf{H}^2(\mathbb{U})$ is an outer function if and only if there holds

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f^*(e^{it})|dt. \tag{2.7}$$

For a bounded outer function $f \in \mathbf{H}^2(\mathbb{U})$, we denote by $\mathbf{H}^2_f(\mathbb{U})$ the completion of the linear space of the functions in $\mathbf{H}^2(\mathbb{U})$ endowed with the inner product

$$\langle g, h\rangle_{\mathbf{H}^2_f(\mathbb{U})} := \frac{1}{2\pi}\int_0^{2\pi} g^*(e^{it})\overline{h^*(e^{it})}|f^*(e^{it})|^2dt, \quad g, h \in \mathbf{H}^2_f(\mathbb{U}). \tag{2.8}$$

Clearly, $\mathbf{H}^2_f(\mathbb{U})$ is a Hilbert space. It is shown in Ref. 7 that if $e_j \in \mathbf{H}^2(\mathbb{U})$, $j \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_f(\mathbb{U})$, then fe_j, $j \in \mathbb{Z}_+$, constitute an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$.

By the general construction described above, we shall present a specific class of orthonormal bases for $\mathbf{H}^2(\mathbb{U})$. Throughout this paper, we denote by a an arbitrary constant in $\mathbb{U}$ with $a = |a|e^{it_a}$, where $t_a \in [0, 2\pi)$. For such an a, we set

$$g(z) := \sqrt{1-|a|^2}\frac{1}{1-\bar{a}z}, \quad z \in \mathbb{U}. \tag{2.9}$$

Thus, $g \in \mathbf{H}^2(\mathbb{U})$. We next show that g is a bounded outer function. For this purpose, we recall the Jensen formula (see, Ref. 13, p. 59).

Lemma 2.1. *For $0 < r < R$, if g is analytic in $\{z \in \mathbb{C} : |z| < R\}$ and has no zeros in $\{z \in \mathbb{C} : |z| \le r\}$ then*

$$\frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\theta})|d\theta = \log|g(0)|. \tag{2.10}$$

Proposition 2.2. *The function g defined by* (2.9) *is a bounded outer function in* $\mathbf{H}^2(\mathbb{U})$.

Proof. When $a = 0$, g is a constant function and is apparently a bounded outer function. Assume that $a \in \mathbb{U}\backslash\{0\}$. First observe that

$$|g(z)|^2 \le \frac{1+|a|}{1-|a|}, \quad z \in \mathbb{U}.$$

Moreover, it is obvious that g satisfies the assumptions of Lemma 2.1. By the Jensen formula, there holds (2.7), which implies that g is an outer function. □

It was pointed out in Ref. 9 that $(\frac{z-a}{1-\bar{a}z})^n$, $z \in \mathbb{U}$, for $n \in \mathbb{Z}_+$, form an orthonormal basis for $\mathbf{H}^2_g(\mathbb{U})$ related to the inner product (2.8). By the general construction described earlier, we conclude that functions

$$\varphi_n(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}\left(\frac{z-a}{1-\bar{a}z}\right)^n, \quad z \in \mathbb{U}, \quad n \in \mathbb{Z}_+ \tag{2.11}$$

constitute an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$.

To obtain orthonormal bases for $L^2_r[0, 2\pi]$ from the orthonormal bases given above, we shall need the following lemma.

Lemma 2.3. *Functions $m_j, j \in \mathbb{Z}_+$ form an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$ if and only if $1, zm_j, j \in \mathbb{Z}_+$ constitute an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$.*

Proof. If for each $j \in \mathbb{Z}_+$, $m_j \in \mathbf{H}^2(\mathbb{U})$, then it has the form

$$m_j(z) = \sum_{n\in\mathbb{Z}_+} a_n^{(j)} z^n, \quad z \in \mathbb{U},$$

where $\{a_n^{(j)} : n \in \mathbb{Z}_+\}$ satisfies $\sum_{n\in\mathbb{Z}_+} |a_n^{(j)}|^2 < \infty$. This implies that $zm_j \in \mathbf{H}^2(\mathbb{U})$, $j \in \mathbb{Z}_+$. On the other hand, the assumption that for each $j \in \mathbb{Z}_+$, $zm_j \in \mathbf{H}^2(\mathbb{U})$ is orthogonal to 1 yields that

$$zm_j(z) = \sum_{n\in\mathbb{N}} b_n^{(j)} z^n, \quad z \in \mathbb{U},$$

with $\sum_{n\in\mathbb{N}} |b_n^{(j)}|^2 < \infty$. Hence, we have that $m_j \in \mathbf{H}^2(\mathbb{U})$, $j \in \mathbb{Z}_+$.

By the definition of the inner product of the Hardy space $\mathbf{H}^2(\mathbb{U})$, we find that for each $j, k \in \mathbb{Z}_+$

$$\langle 1, zm_j\rangle_{\mathbf{H}^2(\mathbb{U})} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n\in\mathbb{Z}_+} \overline{a_n^{(j)}} e^{-i(n+1)t}\right) dt = 0,$$

and $\langle zm_j, zm_k\rangle_{\mathbf{H}^2(\mathbb{U})} = \langle m_j, m_k\rangle_{\mathbf{H}^2(\mathbb{U})}$. Therefore, we conclude that the orthonormality of the sequence m_j, $j \in \mathbb{Z}_+$ is equivalent to that of $1, zm_j$, $j \in \mathbb{Z}_+$.

On the other hand, we can obtain that

$$\overline{\text{span}}\{m_j : j \in \mathbb{Z}_+\} = \overline{\text{span}}\{z^j : j \in \mathbb{Z}_+\},$$

holds if and only if

$$\overline{\text{span}}\{1, zm_j : j \in \mathbb{Z}_+\} = \overline{\text{span}}\{1, zz^j : j \in \mathbb{Z}_+\}.$$

It implies the equivalence of the density of the two bases. □

We let σ_a be the real-valued function on $[0, 2\pi]$ defined by

$$e^{i\sigma_a(t)} = \frac{e^{it}-a}{1-\bar{a}e^{it}}, \quad t \in [0, 2\pi]. \tag{2.12}$$

It has a positive derivative

$$\frac{d\sigma_a(t)}{dt} = \frac{1-|a|^2}{|e^{it}-a|^2}, \quad t \in [0, 2\pi].$$

The function σ_a may be continuously extended to $\mathbb{R}$ with the property $\sigma_a(t+2\pi) = \sigma_a(t)+2\pi$. We also denote the extended function by σ_a. Since σ_a is strictly monotone on $\mathbb{R}$, the inverse σ_a^{-1} exists and satisfies $\sigma_a^{-1}(s+2\pi) = \sigma_a^{-1}(s) + 2\pi,\ s \in \mathbb{R}$. Consequently, we get the following desired orthonormal basis for $L_r^2[0, 2\pi]$.

Theorem 2.4. *The sequence of functions*

$$\sqrt{\frac{1}{2\pi}}, \quad \sqrt{\frac{1}{\pi}}\rho_n \cos\theta_n, \quad \sqrt{\frac{1}{\pi}}\rho_n \sin\theta_n, \quad n \in \mathbb{Z}_+, \tag{2.13}$$

form an orthonormal basis for $L_r^2[0, 2\pi]$, where for $n \in \mathbb{Z}_+$ and $t \in [0, 2\pi]$

$$\rho_n(t)\cos\theta_n(t) = \frac{\sqrt{1-|a|^2}}{1-2\mathrm{Re}(ae^{-it})+|a|^2}(\cos(n\sigma_a(t)+t) - |a|\cos(n\sigma_a(t)+t_a)) \tag{2.14}$$

and

$$\rho_n(t)\sin\theta_n(t) = \frac{\sqrt{1-|a|^2}}{1-2\mathrm{Re}(ae^{-it})+|a|^2}(\sin(n\sigma_a(t)+t) - |a|\sin(n\sigma_a(t)+t_a)). \tag{2.15}$$

Moreover, every nonconstant function in the sequence has positive instantaneous frequency.

Proof. Since for $a \in \mathbb{U}$, the functions $\varphi_n,\ n \in \mathbb{Z}_+$, defined by (2.11) form an orthonormal basis for the Hardy space $\mathbf{H}^2(\mathbb{U})$, by Lemma 2.3, $\{1\} \cup \{z\varphi_n : n \in \mathbb{Z}_+\}$ constitutes an orthonormal basis for $\mathbf{H}^2(\mathbb{U})$. Since for each $n \in \mathbb{Z}_+$, $\rho_n \cos\theta_n$, and $\rho_n \sin\theta_n$ are, respectively, the real and imaginary parts of the nontangential limits of $z\varphi_n$, the system of functions defined by (2.13) forms an orthonormal basis for $L_r^2[0, 2\pi]$. Moreover, for each $n \in \mathbb{Z}_+$ we have that

$$\theta_n(t) = n\sigma_a(t) + \arctan\frac{\sin(t-t_a)}{\cos(t-t_a)-|a|} + t_a, \quad t \in [0, 2\pi].$$

The fact that σ_a has a positive derivative yields $\theta_n' > 0,\ n \in \mathbb{Z}_+$. Consequently, the function system (2.13) forms an orthonormal basis with positive instantaneous frequency for $L_r^2[0, 2\pi]$. □

Theorem 2.4 gives a class of orthonormal bases for $L_r^2[0, 2\pi]$. We will call each of them an A-system. We remark that the A-systems contain the classical real Fourier basis as a special example, which corresponds to the case when $a = 0$. Moreover, an A-system has a positive nonconstant instantaneous frequency for each $a \neq 0$. The parameter a in the A-systems allows us to choose an orthonormal basis for $L_r^2[0, 2\pi]$ adapted to the given data.

3. Properties of the A-System

In this section, we study the zero and extrema properties of the A-system. Through a change of variables $s = \sigma_a(t)$, we have for $n \in \mathbb{Z}_+$ and $t \in [0, 2\pi]$ that

$$\sqrt{\frac{1}{\pi}}\rho_n(t)\cos\theta_n(t) = \frac{1}{\sqrt{\pi(1-|a|^2)}}(\cos(n+1)s + |a|\cos(ns + t_a))$$

and

$$\sqrt{\frac{1}{\pi}}\rho_n(t)\sin\theta_n(t) = \frac{1}{\sqrt{\pi(1-|a|^2)}}(\sin(n+1)s + |a|\sin(ns + t_a)).$$

We introduce two sequences of functions

$$f_n(s) := \cos(n+1)s + |a|\cos(ns + t_a), \quad s \in [0, 2\pi] \tag{3.1}$$

and

$$g_n(s) := \sin(n+1)s + |a|\sin(ns + t_a), \quad s \in [0, 2\pi]. \tag{3.2}$$

Since σ_a is strictly monotone, it suffices to study the zero and extrema properties of f_n and g_n, $n \in \mathbb{Z}_+$. To this end, for $c \in \mathbb{U}$ and $n \in \mathbb{Z}_+$, we define two particular polynomials $P_{n,c}$ and $Q_{n,c}$ by

$$P_{n,c}(z) := z^{2n+2} + cz^{2n+1} + \bar{c}z + 1, \quad Q_{n,c}(z) := z^{2n+2} + cz^{2n+1} - \bar{c}z - 1, \quad z \in \mathbb{C},$$

and consider their zeros. We need the Möbius transform defined for $c \in \mathbb{U}$ by

$$M_c(z) = \frac{z+c}{1+\bar{c}z}, \quad z \in \mathbb{C}. \tag{3.3}$$

It is well known that $|M_c(z)| < 1$ for $|z| < 1$ and $|M_c(z)| > 1$ for $|z| > 1$.

Lemma 3.1. *Both $P_{n,c}$ and $Q_{n,c}$ have exactly $2n+2$ distinct zeros on $\partial\mathbb{U}$ for all $c \in \mathbb{U}$ and $n \in \mathbb{Z}_+$.*

Proof. We present a proof for $P_{n,c}$ only since that for $Q_{n,c}$ is similar. For each $n \in \mathbb{Z}_+$, as a polynomial of degree $2n+2$, $P_{n,c}$ has exactly $2n+2$ zeros in $\mathbb{C}$. Note that $z = 0$ is not a zero of $P_{n,c}$. Suppose that z_0 is a zero of $P_{n,c}$. We then see that

$$|M_c(z_0)| = \frac{1}{|z_0|^{2n+1}}, \tag{3.4}$$

where M_c is the Möbius transform as (3.3). This fact with the property of the Möbius transform yields that z_0 in (3.4) must be of modulus 1. Therefore, P_n has exactly $2n+2$ zeros on $\partial\mathbb{U}$.

We next show that any zero of $P_{n,c}$ is simple. If $P_{n,c}(z_0) = P'_{n,c}(z_0) = 0$ for some $z_0 \in \partial\mathbb{U}$, we will get the equations

$$z_0^{2n+2} + cz_0^{2n+1} + \bar{c}z_0 + 1 = 0 \tag{3.5}$$

and

$$(2n+2)z_0^{2n+2} + c(2n+1)z_0^{2n+1} + \bar{c}z_0 = 0. \tag{3.6}$$

Subtracting (3.5) from (3.6), we obtain that $(2n+1)z_0^{2n+2} = 1 - 2ncz_0^{2n}$. Together with the fact that $z_0 \in \partial\mathbb{U}$, this equation yields that $2n+1 \le 2n|c|+1$, which is impossible since $|c|<1$. This contradiction shows that every zero of $P_{n,c}$ is simple. It follows that the zeros of $P_{n,c}$ are pair-wise distinct. □

We now return to the study of the zero property of functions f_n and g_n.

Theorem 3.2. *The following statements hold true*:

(i) *For each $n \in \mathbb{Z}_+$, f_n has exactly $2n+2$ distinct zeros in $[0, 2\pi]$.*
(ii) *For each $n \in \mathbb{Z}_+$, g_n has exactly $2n+3$ distinct zeros in $[0, 2\pi]$ if $a \in (-1, 1)$ and g_n has exactly $2n+2$ distinct zeros in $[0, 2\pi]$ if $a \notin (-1, 1)$.*

Proof. (1) For each $n \in \mathbb{Z}_+$, we introduce the function

$$F_n(z) := \frac{1}{2}\left(z^{n+1} + az^n + \bar{a}\frac{1}{z^n} + \frac{1}{z^{n+1}}\right), \quad z \in \mathbb{C}\backslash\{0\},$$

and observe that the function f_n has the form $f_n(s) = F_n(e^{is})$. Thus, we have that $s \in [0, 2\pi]$ is a zero of f_n of order m if and only if $z := e^{is}$ is a zero of F_n of the same order. Note that for $z \in \mathbb{C}\backslash\{0\}$

$$z^{n+1}F_n(z) = \frac{1}{2}P_{n,a}(z).$$

Then it is clear that for $z \neq 0$, z is a zero of F_n of order m if and only if it is a zero of $P_{n,a}$ of the same order. Since $z = 1$ is not a zero of $P_{n,a}$, to prove that f_n has exactly $2n+2$ distinct zeros in $[0, 2\pi]$, it suffices to show that $P_{n,a}$ has exactly $2n+2$ distinct zeros in $\partial\mathbb{U}$. This is confirmed by Lemma 3.1.

(2) For each $n \in \mathbb{Z}_+$, we let

$$G_n(z) := \frac{1}{2i}\left(z^{n+1} + az^n - \bar{a}\frac{1}{z^n} - \frac{1}{z^{n+1}}\right), \quad z \in \mathbb{C}\backslash\{0\},$$

and observe that the function g_n has the form $g_n(s) = G_n(e^{is})$ and

$$z^{n+1}G_n(z) = \frac{1}{2i}Q_{n,a}(z), \quad z \in \mathbb{C}\backslash\{0\}.$$

We have that $s \in [0, 2\pi]$ is a zero of g_n of order m if and only if $z = e^{is}$ is a zero of $Q_{n,a}$ of the same order. If $a \in (-1, 1)$, then $z = 1$ is a zero of $Q_{n,a}$. Hence the fact $e^{i\cdot 0} = e^{i\cdot 2\pi} = 1$ shows that $s = 0$ and $s = 2\pi$ are both zeros of g_n. However, any other zero of $Q_{n,a}$ on $\partial\mathbb{U}$ corresponds to only one zero of g_n in $[0, 2\pi]$. Combining these with the fact that $Q_{n,a}$ has $2n+2$ distinct zeros on $\partial\mathbb{U}$, we have g_n has exactly $2n+3$ distinct zeros in $[0, 2\pi]$. Otherwise, $z = 1$ is not a zero of $Q_{n,a}$ and thus g_n has exactly $2n+2$ distinct zeros in $[0, 2\pi]$ as $Q_{n,a}$ does on $\partial\mathbb{U}$. □

For the extrema property of functions f_n and g_n, we have the following result.

Theorem 3.3. *The following statements hold true*:

(i) *For each* $n \in \mathbb{Z}_+$, f_n *has exactly* $2n+1$ *extrema in* $[0, 2\pi]$ *if* $a \in (-1, 1)$ *and* f_n *has exactly* $2n+2$ *extrema in* $[0, 2\pi]$ *if* $a \notin (-1, 1)$.
(ii) *For each* $n \in \mathbb{Z}_+$, g_n *has exactly* $2n+2$ *extrema in* $[0, 2\pi]$.

Proof. For each $n \in \mathbb{Z}_+$, there hold

$$f_n'(s) = -(n+1)\left(\sin(n+1)s + \frac{n}{n+1}|a|\sin(ns+t_a)\right), \quad s \in [0, 2\pi]$$

and

$$g_n'(s) = (n+1)\left(\cos(n+1)s + \frac{n}{n+1}|a|\cos(ns+t_a)\right), \quad s \in [0, 2\pi].$$

Let $c := \frac{n}{n+1}a$. Then f_n' and g_n' can be rewritten, respectively, as

$$f_n'(s) = -(n+1)(\sin(n+1)s + |c|\sin(ns+t_c)), \quad s \in [0, 2\pi]$$

and

$$g_n'(s) = (n+1)(\cos(n+1)s + |c|\cos(ns+t_c)), \quad s \in [0, 2\pi].$$

By Theorem 3.2 and the fact that $c \in \mathbb{U}$, we have f_n' has exactly $2n+3$ distinct zeros in $[0, 2\pi]$ if $c \in (-1, 1)$ and f_n' has exactly $2n+2$ distinct zeros in $[0, 2\pi]$ if $c \notin (-1, 1)$. Since both 0 and 2π are the zeros of f_n' if $c \in (-1, 1)$ and neither 0 or 2π is the zeros of f_n' if $c \notin (-1, 1)$, we obtain that f_n' has exactly $2n+1$ distinct zeros in $(0, 2\pi)$ if $c \in (-1, 1)$ and f_n' has exactly $2n+2$ distinct zeros in $(0, 2\pi)$ if $c \notin (-1, 1)$. Likewise, g_n' has exactly $2n+2$ distinct zeros in $(0, 2\pi)$. Consequently, since any zero of f_n' or g_n' is simple, we can get the conclusion that f_n has $2n+1$ extrema in $[0, 2\pi]$ if $a \in (-1, 1)$ and $2n+2$ extrema in $[0, 2\pi]$ if $a \notin (-1, 1)$ and g_n has $2n+2$ extrema in $[0, 2\pi]$. $\square$

In the following, we show that for f_n, g_n, $n \in \mathbb{Z}_+$, the zeros and the extrema interlace. To this end, we present two technical lemmas.

Lemma 3.4. *Let* $\alpha, \beta \in \mathbb{R}$. *If* $\sin\alpha + c_1 \sin\beta = 0$ *for some* $c_1 \in [0, 1)$ *then for all* $c_2, c_3 \in [0, 1)$,

$$(\cos\alpha + c_2\cos\beta)(\cos\alpha + c_3\cos\beta) > 0.$$

Proof. Direct computations show that

$$\begin{aligned}
&(\cos\alpha + c_2\cos\beta)(\cos\alpha + c_3\cos\beta)\\
&\quad = \cos^2\alpha + c_2c_3\cos^2\beta + (c_2+c_3)\cos\alpha\cos\beta\\
&\quad \geq \cos^2\alpha + c_2c_3\cos^2\beta - \frac{c_2+c_3}{2}(\cos^2\alpha + \cos^2\beta)\\
&\quad = \left(1 - \frac{c_2+c_3}{2}\right)\cos^2\alpha + \left(c_2c_3 - \frac{c_2+c_3}{2}\right)\cos^2\beta.
\end{aligned}$$

The assumption $\sin\alpha + c_1 \sin\beta = 0$ implies that $\cos^2\alpha = 1 - c_1^2 + c_1^2 \cos^2\beta$. This combined with the above inequality yields that

$$\begin{aligned}(\cos\alpha + c_2\cos\beta)&(\cos\alpha + c_3\cos\beta)\\ &\geq \left(1 - \frac{c_2+c_3}{2}\right)(1 - c_1^2 + c_1^2\cos^2\beta) + \left(c_2c_3 - \frac{c_2+c_3}{2}\right)\cos^2\beta. \qquad (3.7)\end{aligned}$$

Because the right-hand side of the above inequality is a linear function of $\cos^2\beta$, it suffices to show that it is positive at the two endpoints $\cos^2\beta = 0$ and $\cos^2\beta = 1$. When $\cos^2\beta = 0$, the right-hand side of (3.7) becomes $(1 - \frac{c_2+c_3}{2})(1 - c_1^2) > 0$. When $\cos^2\beta = 1$, it is equal to $1 - (c_2 + c_3) + c_2c_3 = (1 - c_2)(1 - c_3) > 0$. This completes the proof of this lemma. □

Arguments similar to those used in the proof of the last lemma prove the following result.

Lemma 3.5. *Let* $\alpha, \beta \in \mathbb{R}$. *If* $\cos\alpha + c_1\cos\beta = 0$ *for some* $c_1 \in [0, 1)$ *then for all* $c_2, c_3 \in [0, 1)$,

$$(\sin\alpha + c_2\sin\beta)(\sin\alpha + c_3\sin\beta) > 0.$$

With the above two lemmas, we establish the next theorem.

Theorem 3.6. *For each* $n \in \mathbb{Z}_+$, *the zeros and the extrema of* f_n *(resp.* g_n*) interlace.*

Proof. We first prove that f_n has at least one extrema between its two consecutive zeros. We suppose that s_1 and s_2 are two consecutive zeros of f_n. Then there exists $s_0 \in (s_1, s_2)$ such that $f_n'(s_0) = 0$. Since all the zeros of f_n' are simple, we conclude that $f_n''(s_0) \neq 0$ and so s_0 is a extrema of f_n. It suffices to prove that there exists at least one zero of f_n between its two consecutive extrema. Assume s_1 and s_2 are two successive extrema of f_n. Then there hold

$$\sin(n+1)s_1 + \frac{n}{n+1}|a|\sin(ns_1 + t_a) = 0 \qquad (3.8)$$

and

$$\sin(n+1)s_2 + \frac{n}{n+1}|a|\sin(ns_2 + t_a) = 0. \qquad (3.9)$$

Since f_n' has only simple zeros and s_1, s_2 are two consecutive extrema of f_n, we also have that $f_n''(s_1)f_n''(s_2) < 0$, where

$$f_n''(s_1) = -(n+1)^2\left(\cos(n+1)s_1 + \left(\frac{n}{n+1}\right)^2|a|\cos(ns_1 + t_a)\right) \qquad (3.10)$$

and

$$f_n''(s_2) = -(n+1)^2\left(\cos(n+1)s_2 + \left(\frac{n}{n+1}\right)^2|a|\cos(ns_2 + t_a)\right). \qquad (3.11)$$

Let $c_1 := \frac{n}{n+1}|a|$, $c_2 := (\frac{n}{n+1})^2|a|$, $c_3 := |a|$. Then, Eq. (3.8) shows that $\sin\alpha + c_1 \sin\beta = 0$, where $\alpha = (n+1)s_1$ and $\beta = ns_1 + t_a$. By Lemma 3.4, we have that $f_n''(s_1)f_n(s_1) < 0$. Likewise, Eq. (3.9) implies that $f_n''(s_2)f_n(s_2) < 0$. Together with the fact that $f_n''(s_1)f_n''(s_2) < 0$, these yield that $f_n(s_1)f_n(s_2) < 0$ and so there exists $s_0 \in (s_1, s_2)$ such that $f_n(s_0) = 0$. For the function g_n, the result can be similarly proved by Lemma 3.5. □

Finally, we have the desirable property of the zeros and the extrema of the basis functions constructed in Proposition 2.4.

Theorem 3.7. *The zeros and the extrema of the basis functions in the A-system interlace for all $a \in \mathbb{U}$.*

Theorem 3.7 shows that the basis functions we have constructed satisfy the first condition of IMF in EMD algorithm. A real function with the condition has been characterized mathematically as a solution of a self-adjoint ordinary differential equation (cf. Ref. 14).

4. The A-Conjugate Pair

Our next task is to establish a fast algorithm to decompose a function in $L_r^2[0, 2\pi]$ as a sum of an A-system. To this end, in this section we introduce the A-conjugate pair of a function and present its properties.

Definition 4.1. For each $f \in L^1[0, 2\pi]$, the functions $f_{a,1}$ and $f_{a,2}$ defined by

$$f_{a,1}(s) = \sqrt{1-|a|^2}(f \circ \sigma_a^{-1})(s)\frac{1}{e^{is}+a}, \quad s \in [0, 2\pi] \tag{4.1}$$

and

$$f_{a,2}(s) = \sqrt{1-|a|^2}(f \circ \sigma_a^{-1})(s)\frac{1}{e^{-is}+\bar{a}}, \quad s \in [0, 2\pi], \tag{4.2}$$

are called an A-conjugate pair of f.

In the following, we show that the A-conjugate pair of f has the same properties as those of f.

Lemma 4.2. *If $f \in L^p[0, 2\pi]$, $1 \le p < \infty$, then $f_{a,k} \in L^p[0, 2\pi]$ for $k = 1, 2$.*

Proof. The facts that $\sigma_a^{-1}(s+2\pi) = \sigma_a^{-1}(s) + 2\pi$, $s \in \mathbb{R}$ and f is 2π-periodic yield that $f_{a,k}$, $k = 1, 2$ are 2π-periodic. By the definition of $f_{a,k}$, $k = 1, 2$, we have that

$$\int_0^{2\pi} |f_{a,k}(s)|^p ds = (1-|a|^2)^{\frac{p}{2}} \int_0^{2\pi} |(f \circ \sigma_a^{-1})(s)|^p \left|\frac{1}{e^{is}+a}\right|^p ds.$$

By a change of variables $s = \sigma_a(t)$, we obtain that

$$\int_0^{2\pi} |f_{a,k}(s)|^p ds = (1-|a|^2)^{1-\frac{p}{2}} \int_0^{2\pi} |f(t)|^p |e^{it}-a|^{p-2} dt.$$

The result of this lemma follows from the observation that $1-|a| \le |e^{it}-a| \le 1+|a|$. □

Lemma 4.3. *If f is a 2π-periodic function with m successive absolutely continuous derivatives then so is $f_{a,k}$ for each $k = 1, 2$.*

Proof. The result follows immediately from the fact that both the composition and product of two absolutely continuous functions are absolutely continuous. □

We next consider the bounded variation property for the conjugate pair. For each function f on $[0, 2\pi]$ we denote by $V(f)$ its total variation defined by

$$V(f) := \sup \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|,$$

where the supremum is taken over all n and all choices of $\{x_j\}$ such that $0 = x_0 < x_1 < \cdots < x_n = 2\pi$.

Lemma 4.4. *If f is 2π-periodic and has bounded variation on* $[0, 2\pi]$, *then so is $f_{a,k}$ for each $k = 1, 2$. Moreover, there holds*

$$V(f_{a,k}) \leq \sqrt{\frac{1+|a|}{1-|a|}} \left(V(f) + \frac{2\pi}{1-|a|} \|f\|_{L^\infty[0,2\pi]} \right), \quad k = 1, 2. \tag{4.3}$$

Proof. Since σ_a^{-1} is strictly monotone and it maps $[0, 2\pi]$ onto a closed interval with length 2π, any partition $0 = x_0 < x_1 < \cdots < x_n = 2\pi$ of $[0, 2\pi]$ corresponds to a partition of the closed interval. Note that f has bounded variation on any closed interval with length 2π. Hence, we conclude that $f \circ \sigma_a^{-1}$ has bounded variation on $[0, 2\pi]$ and $V(f) = V(f \circ \sigma_a^{-1})$. It is clear that $\frac{1}{e^{i\cdot}+a}$ has bounded variation on $[0, 2\pi]$ and $V(\frac{1}{e^{i\cdot}+a}) \leq \frac{2\pi}{(1-|a|)^2}$. For any partition $0 = x_0 < x_1 < \cdots < x_n = 2\pi$, we observe that for $k = 1, 2$ that

$$\begin{aligned}
&\sum_{j=1}^{n} |f_{a,k}(x_j) - f_{a,k}(x_{j-1})| \\
&\qquad \leq \sqrt{1-|a|^2} \sum_{j=1}^{n} |(f \circ \sigma_a^{-1})(x_j) - (f \circ \sigma_a^{-1})(x_{j-1})| \frac{1}{|e^{ix_j} + a|} \\
&\qquad\quad + \sqrt{1-|a|^2} \sum_{j=1}^{n} \left| \frac{1}{e^{ix_j} + a} - \frac{1}{e^{ix_{j-1}} + a} \right| |(f \circ \sigma_a^{-1})(x_{j-1})|.
\end{aligned}$$

Taking the supremum over all the partitions yields that

$$V(f_{a,k}) \leq \sqrt{1-|a|^2} \left(V(f) \left\| \frac{1}{e^{i\cdot} + a} \right\|_{L^\infty[0,2\pi]} + \|f\|_{L^\infty[0,2\pi]} V\left(\frac{1}{e^{i\cdot} + a} \right) \right),$$

which implies (4.3) and completes the proof. □

5. Nonlinear Fourier Series

In this section, we introduce nonlinear Fourier series which we call the Hilbert–Fourier series and establish the relationship between the Hilbert–Fourier coefficients of $f \in L^1[0, 2\pi]$ and the classical Fourier coefficients of the A-conjugate pair of f. Moreover, we investigate the properties of the Hilbert–Fourier series and Hilbert–Fourier coefficients.

By the A-system, functions

$$\sqrt{\frac{1}{2\pi}}, \quad \sqrt{\frac{1}{2\pi}}\rho_n e^{i\theta_n}, \quad \sqrt{\frac{1}{2\pi}}\rho_n e^{-i\theta_n}, \quad n \in \mathbb{Z}_+,$$

form an orthonormal basis for $L^2[0, 2\pi]$, where for $n \in \mathbb{Z}_+$

$$\rho_n(t)e^{i\theta_n(t)} = \sqrt{1-|a|^2}\frac{e^{it}}{1-\bar{a}e^{it}}\left(\frac{e^{it}-a}{1-\bar{a}e^{it}}\right)^n, \quad t \in [0, 2\pi].$$

We denote by e_n^a, $n \in \mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2 \ldots\}$, the Hilbert–Fourier basis related to a, which are defined by

$$e_n^a = \begin{cases} \rho_{n-1}e^{i\theta_{n-1}}, & n = 1, 2, \ldots, \\ 1, & n = 0, \\ \rho_{-n-1}e^{-i\theta_{-n-1}}, & n = -1, -2, \ldots. \end{cases} \tag{5.1}$$

For each $f \in L^1[0, 2\pi]$, the coefficients of the series

$$\sum_{n\in\mathbb{Z}} c_n^a(f)e_n^a(t), \quad t \in [0, 2\pi] \tag{5.2}$$

are defined by

$$c_n^a(f) := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{e_n^a(t)}dt. \tag{5.3}$$

The series (5.2) is called the Hilbert–Fourier series of f and coefficient (5.3) is called the nth Hilbert–Fourier coefficient of f.

The following theorem connects the Hilbert–Fourier coefficients with the classical Fourier coefficients.

Theorem 5.1. *If $f \in L^1[0, 2\pi]$ then the Hilbert–Fourier coefficients of f are determined by*

$$c_n^a(f) = \begin{cases} c_{n-1}(f_{a,1}), & n = 1, 2, \ldots, \\ c_0(f), & n = 0, \\ c_{n+1}(f_{a,2}), & n = -1, -2, \ldots, \end{cases} \tag{5.4}$$

where $c_n(g)$ denotes the nth classical Fourier coefficient of function g.

Proof. By Lemma 4.2, $f_{a,k} \in L^1[0, 2\pi]$ for $k = 1, 2$. By definition we have for $n \in \mathbb{N}$

$$c_n^a(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\sqrt{1-|a|^2}\frac{e^{-it}}{1-ae^{-it}}\left(\frac{e^{it}-a}{1-\bar{a}e^{it}}\right)^{-n+1} dt$$

and for $-n \in \mathbb{N}$

$$c_n^a(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\sqrt{1-|a|^2}\frac{e^{it}}{1-\bar{a}e^{it}}\left(\frac{e^{it}-a}{1-\bar{a}e^{it}}\right)^{-n-1} dt.$$

By a change of variables $s = \sigma_a(t)$, we obtain that for $n \in \mathbb{N}$

$$c_n^a(f) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-|a|^2}(f\circ\sigma_a^{-1})(s)\frac{1}{e^{is}+a}e^{-i(n-1)s}ds = c_{n-1}(f_{a,1})$$

and for $-n \in \mathbb{N}$

$$c_n^a(f) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-|a|^2}(f\circ\sigma_a^{-1})(s)\frac{1}{e^{-is}+\bar{a}}e^{-i(n+1)s}ds = c_{n+1}(f_{a,2}).$$

Finally, it is clear that

$$c_0^a(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = c_0(f).$$

The proof is complete. □

By Theorem 5.1 and the Riemmann–Lebesgue lemma, we have the following decay properties of the Hilbert–Fourier coefficients.

Corollary 5.2. *If* $f \in L^1[0,2\pi]$ *then* $\lim_{n\to\pm\infty} c_n^a(f) = 0$.

If we impose some regularity conditions upon f then we can further estimate the decay rate of the Hilbert–Fourier coefficients. For the classical Fourier coefficients, it is well known that if f is 2π-periodic with $m-1$ successive absolutely continuous derivatives then its Fourier coefficients satisfy

$$c_n(f) = o\left(\frac{1}{|n|^m}\right), \quad |n|\to\infty. \tag{5.5}$$

We have a similar result for the Hilbert–Fourier coefficients.

Corollary 5.3. *If* f *is* 2π*-periodic with* $m-1$ *successive absolutely continuous derivatives then*

$$c_n^a(f) = o\left(\frac{1}{|n|^m}\right), \quad |n|\to\infty. \tag{5.6}$$

Proof. By Lemma 4.4, for each $k = 1, 2$, $f_{a,k}$ is 2π-periodic with $m-1$ successive absolutely continuous derivatives. Thus Eqs. (5.4) and (5.5) lead to

$$c_n^a(f) = c_{n-1}(f_{a,1}) = o\left(\frac{1}{|n-1|^m}\right), \quad n\to\infty$$

and

$$c_n^a(f) = c_{n+1}(f_{a,2}) = o\left(\frac{1}{|n+1|^m}\right), \quad n \to -\infty,$$

which imply the estimation (5.6). □

It is well known that if f is a 2π-periodic and has bounded variation on $[0, 2\pi]$ then its Fourier coefficients satisfy

$$|c_n(f)| \leq \frac{1}{4|n|} V(f), \quad n \in \mathbb{Z}\backslash\{0\}. \tag{5.7}$$

Similarly, the Hilbert–Fourier coefficients of a function of bounded variation have the following decay rate.

Corollary 5.4. *If f is 2π-periodic and has bounded variation on $[0, 2\pi]$ then for $|n| \geq 2$*

$$|c_n^a(f)| \leq \sqrt{\frac{1+|a|}{1-|a|}} \frac{1}{4(|n|-1)} \left(V(f) + \frac{2\pi}{1-|a|} \|f\|_{L^\infty[0,2\pi]}\right). \tag{5.8}$$

Proof. By Eqs. (5.7) and (5.4), we have the following estimations

$$|c_n^a(f)| = |c_{n-1}(f_{a,1})| \leq \frac{1}{4(|n|-1)} V(f_{a,1}), \quad n \geq 2$$

and

$$|c_n^a(f)| = |c_{n+1}(f_{a,2})| \leq \frac{1}{4(|n|-1)} V(f_{a,2}), \quad n \leq -2.$$

These two equations combined with Lemma 4.4 prove (5.8). □

For the classical Fourier series, Carleson[15] showed that for any $f \in L^2[0, 2\pi]$, its Fourier series converges to f almost everywhere, and Hunt[16] generalized the result to the case $f \in L^p[0, 2\pi]$, $1 < p < \infty$ in 1967. In the following, we establish the almost everywhere convergence of the Hilbert–Fourier series.

Theorem 5.5. *Let $a \in \mathbb{U}$. If $f \in L^p[0, 2\pi]$, $1 < p < \infty$, then its Hilbert–Fourier series defined as in* (5.2) *converges to f almost everywhere on* $[0, 2\pi]$.

Proof. By Eq. (5.4), we have the partial sums of the Hilbert–Fourier series of f as

$$\begin{aligned} S_N^a(t) = &\sum_{n=-N}^{-1} \sqrt{1-|a|^2} c_{n+1}(f_{a,2}) \frac{1}{e^{it}-a} \left(\frac{e^{it}-a}{1-\bar{a}e^{it}}\right)^{n+1} + c_0(f) \\ &+ \sum_{n=1}^{N} \sqrt{1-|a|^2} c_{n-1}(f_{a,1}) \frac{e^{it}}{1-\bar{a}e^{it}} \left(\frac{e^{it}-a}{1-\bar{a}e^{it}}\right)^{n-1}, \quad t \in [0, 2\pi], \end{aligned}$$

where $f_{a,1}$ and $f_{a,2}$ are the A-conjugate pair of f. By a change of variables $s = \sigma_a(t)$, we get that

$$\begin{aligned} S_N^a(\sigma_a^{-1}(s)) = & \frac{1}{\sqrt{1-|a|^2}} c_{-N+1}(f_{a,2}) e^{-iNs} \\ & + \sum_{n=-N+1}^{-1} \frac{1}{\sqrt{1-|a|^2}} (c_{n+1}(f_{a,2}) + \bar{a} c_n(f_{a,2})) e^{ins} \\ & + \frac{\bar{a}}{\sqrt{1-|a|^2}} c_0(f_{a,2}) + c_0(f) + \frac{a}{\sqrt{1-|a|^2}} c_0(f_{a,1}) \\ & + \sum_{n=1}^{N-1} \frac{1}{\sqrt{1-|a|^2}} (c_{n-1}(f_{a,1}) + a c_n(f_{a,1})) e^{ins} \\ & + \frac{1}{\sqrt{1-|a|^2}} c_{N-1}(f_{a,1}) e^{iNs}, \quad s \in [0, 2\pi]. \end{aligned}$$

By a change of variables $s = \sigma_a(t)$, we also obtain that

$$\int_0^{2\pi} |(f \circ \sigma_a^{-1})(s)|^p ds = \int_0^{2\pi} |f(t)|^p \frac{1-|a|^2}{|e^{it}-a|^2} dt \le \frac{1+|a|}{1-|a|} \int_0^{2\pi} |f(t)|^p dt.$$

Together with the fact that $f \in L^p[0, 2\pi]$, this yields that $f \circ \sigma_a^{-1} \in L^p[0, 2\pi]$. Thus, the partial sums of Fourier series of $f \circ \sigma_a^{-1}$

$$\widetilde{S_N}(s) = \sum_{n=-N}^{N} c_n(f \circ \sigma_a^{-1}) e^{ins}, \quad s \in [0, 2\pi],$$

converge to $f \circ \sigma_a^{-1}$ almost everywhere on $[0, 2\pi]$. According to

$$(f \circ \sigma_a^{-1})(s) = \frac{1}{\sqrt{1-|a|^2}} (e^{is} + a) f_{a,1}(s), \quad s \in [0, 2\pi]$$

and

$$(f \circ \sigma_a^{-1})(s) = \frac{1}{\sqrt{1-|a|^2}} (e^{-is} + \bar{a}) f_{a,2}(s), \quad s \in [0, 2\pi],$$

we calculate $c_n(f \circ \sigma_a^{-1})$ and get that

$$\begin{aligned} \widetilde{S_N}(s) = & \sum_{n=-N}^{-1} \frac{1}{\sqrt{1-|a|^2}} (\bar{a} c_n(f_{a,2}) + c_{n+1}(f_{a,2})) e^{ins} + c_0(f \circ \sigma_a^{-1}) \\ & + \sum_{n=1}^{N} \frac{1}{\sqrt{1-|a|^2}} (a c_n(f_{a,1}) + c_{n-1}(f_{a,1})) e^{ins}, \quad s \in [0, 2\pi]. \end{aligned}$$

Hence, for any $s \in [0, 2\pi]$, there holds

$$\begin{aligned} \widetilde{S_N}(s) - S_N^a(\sigma_a^{-1}(s)) = & \frac{1}{\sqrt{1-|a|^2}} (\bar{a} c_{-N}(f_{a,2}) e^{-iNs} + a c_N(f_{a,1}) e^{iNs}) \\ & + c_0(f \circ \sigma_a^{-1}) - c_0(f) - \frac{1}{\sqrt{1-|a|^2}} (\bar{a} c_0(f_{a,2}) + a c_0(f_{a,1})). \end{aligned}$$

By the definition of the A-conjugate pair, we have that

$$\frac{1}{\sqrt{1-|a|^2}}(\bar{a}f_{a,2}(s)+af_{a,1}(s))=\left(1-\frac{1-|a|^2}{|e^{is}+a|^2}\right)(f\circ\sigma_a^{-1})(s),\quad s\in[0,2\pi].$$

This yields that

$$\frac{1}{\sqrt{1-|a|^2}}(\bar{a}c_0(f_{a,2})+ac_0(f_{a,1}))=c_0(f\circ\sigma_a^{-1})-c_0(g),$$

where

$$g(s)=\frac{1-|a|^2}{|e^{is}+a|^2}(f\circ\sigma_a^{-1})(s),\quad s\in[0,2\pi].$$

Again by a change of variables $s=\sigma_a(t)$, we get that $c_0(g)=c_0(f)$. Consequently, for any $s\in[0,2\pi]$, there holds

$$\widetilde{S_N}(s)-S_N^a(\sigma_a^{-1}(s))=\frac{1}{\sqrt{1-|a|^2}}(\bar{a}c_{-N}(f_{a,2})e^{-iNs}+ac_N(f_{a,1})e^{iNs}).\tag{5.9}$$

Combining Corollary 5.2 with Eq. (5.9), we conclude that $S_N^a(\sigma_a^{-1})$ converges to $f\circ\sigma_a^{-1}$ almost everywhere on $[0,2\pi]$. The fact that σ_a is strictly increasing yields that S_N^a converges to f almost everywhere on $[0,2\pi]$. □

For the pointwise convergence for the Fourier series, the Jordan criteria was given in Ref. 17.

Lemma 5.6. *If $f\in L^1[0,2\pi]$ is of bounded variation on a neighborhood $[t-\delta,t+\delta]$ of $t\,(\delta>0)$, then its Fourier series converges to $\frac{1}{2}[f(t+0)+f(t-0)]$ at t. If, in addition, f is continuous at t, then its Fourier series converges to f at t.*

The following theorem gives an sufficient condition for pointwise convergence of Hilbert–Fourier series.

Theorem 5.7. *If $f\in L^1[0,2\pi]$ is of bounded variation on a neighborhood $[t-\delta,t+\delta]$ of $t\,(\delta>0)$, then its Hilbert–Fourier series converges to $\frac{1}{2}[f(t+0)+f(t-0)]$ at t. If, in addition, f is continuous at t, then its Hilbert–Fourier series converges to f at t.*

Proof. Since σ_a is strictly monotone, the assumption that f is of bounded variation on some neighborhood of t yields that $f\circ\sigma_a^{-1}$ is of bounded variation on some neighborhood of $\sigma_a(t)$. Hence, by Lemma 5.6, we have that the partial sums of the Fourier series of $f\circ\sigma_a^{-1}$ at the point $s=\sigma_a(t)$

$$\widetilde{S_N}(\sigma_a(t))=\sum_{n=-N}^{N}c_n(f\circ\sigma_a^{-1})e^{in\sigma_a(t)},$$

converges to $\frac{1}{2}[(f\circ\sigma_a^{-1})(\sigma_a(t)+0)+(f\circ\sigma_a^{-1})(\sigma_a(t)-0)]$ when $N\to\infty$. Equation (5.9) shows that the partial sums of the Hilbert–Fourier series of f at the point t satisfies

$$\lim_{N\to\infty} S_N^a(t) = \frac{1}{2}[(f \circ \sigma_a^{-1})(\sigma_a(t)+0) + (f \circ \sigma_a^{-1})(\sigma_a(t)-0)].$$

Combining this with the fact that σ_a is strictly monotone, we have that

$$\lim_{N\to\infty} S_N^a(t) = \frac{1}{2}[f(t+0) + f(t-0)].$$

In addition, if f is continuous at t, we obtain $\lim_{N\to\infty} S_N^a(t) = f(t)$. □

It was shown in Ref. 18 that the classical Fourier basis is a basis for $L^p[0,2\pi],\ 1 < p < \infty$. We next show the similar result for the Hilbert–Fourier basis. Here, we say that a set $\{f_n :\ n \in \mathbb{Z}\} \subset L^p[0,2\pi]$ is a basis for $L^p[0,2\pi],\ 1 < p < \infty$ if for each $f \in L^p[0,2\pi]$ there exists a unique sequence $\alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}$, such that

$$f = \sum_{n\in\mathbb{Z}} \alpha_n f_n,$$

where the convergence of the partial sums is in the norm of $L^p[0,2\pi]$.

Theorem 5.8. *The Hilbert–Fourier basis* $e_n^a,\ n \in \mathbb{Z}$ *is a basis for* $L^p[0,2\pi],\ 1 < p < \infty$.

Proof. We first prove the completeness of $e_n^a,\ n \in \mathbb{Z}$. For any $f \in L^p[0,2\pi]$, we denote the partial sums of its Hilbert–Fourier series by

$$S_N^a(t) = \sum_{n=-N}^{N} c_n^a(f) e_n^a(t), \quad t \in [0,2\pi].$$

Since $f \in L^p[0,2\pi]$ implies $f \circ \sigma_a^{-1} \in L^p[0,2\pi]$, we represent the partial sums of the Fourier series of $f \circ \sigma_a^{-1}$ as

$$\widetilde{S_N}(s) = \sum_{n=-N}^{N} c_n(f \circ \sigma_a^{-1}) e^{ins}, \quad s \in [0,2\pi].$$

According to the proof of Theorem 5.5, formula (5.9) holds for any $s \in [0,2\pi]$. It follows that

$$\begin{aligned}\|f \circ \sigma_a^{-1} - S_N^a \circ \sigma_a^{-1}\|_p &\le \|f \circ \sigma_a^{-1} - \widetilde{S_N}\|_p + \|\widetilde{S_N} - S_N^a \circ \sigma_a^{-1}\|_p \\ &\le \|f \circ \sigma_a^{-1} - \widetilde{S_N}\|_p + \frac{(2\pi)^{\frac{1}{p}}|a|}{\sqrt{1-|a|^2}}(|c_{-N}(f_{a,2})| + |c_N(f_{a,1})|).\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Because $f \circ \sigma_a^{-1} \in L^p[0,2\pi]$, there exists a positive integer $M_1 > 0$, such that $\|f \circ \sigma_a^{-1} - \widetilde{S_N}\|_p < \frac{\epsilon}{2}$ for any positive integer $N > M_1$. On the other hand, combining the fact $f_{a,1}, f_{a,2} \in L^1[0,2\pi]$ with the Riemann–Lebesgue

Lemma, there exists a positive integer $M_2 > 0$ such that for all positive integers $N > M_2$,

$$\frac{(2\pi)^{\frac{1}{p}}|a|}{\sqrt{1-|a|^2}}(|c_{-N}(f_{a2})| + |c_N(f_{a1})|) < \frac{\epsilon}{2}.$$

Thus, for any $\epsilon > 0$, there exists a positive integer $M = \max(M_1, M_2)$, such that for all positive integers $N > M$, there holds $\|f \circ \sigma_a^{-1} - S_N^a \circ \sigma_a^{-1}\|_p < \epsilon$. By a change of variables, we obtain that

$$\|f - S_N^a\|_p \leq \left(\frac{1+|a|}{1-|a|}\right)^{\frac{1}{p}} \|f \circ \sigma_a^{-1} - S_N^a \circ \sigma_a^{-1}\|_p.$$

The completeness is established.

Next for any $f \in L^p[0, 2\pi]$, we show the uniqueness of the representation

$$f(t) = \sum_{n\in\mathbb{Z}} d_n e_n^a(t), \quad t \in [0, 2\pi],$$

which holds in $L^p[0, 2\pi]$. We define the partial sums by

$$\widetilde{S_N^a}(t) = \sum_{n=-N}^{N} d_n e_n^a(t), \quad t \in [0, 2\pi].$$

For $1 < p < \infty$, let q be the conjugate index of p which satisfies $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that $e_n^a \in L^q[0, 2\pi]$. Hence, for any $m \in \mathbb{Z}$, we have that

$$|\langle f - \widetilde{S_N^a}, e_m^a\rangle_{L^2[0,2\pi]}| \leq \|f - \widetilde{S_N^a}\|_p \|e_m^a\|_q.$$

Let $N \to \infty$. Because $\lim_{N\to\infty} \|f - \widetilde{S_N^a}\|_p = 0$, we obtain that

$$\langle f, e_m^a\rangle_{L^2[0,2\pi]} = \lim_{N\to\infty} \langle \widetilde{S_N^a}, e_m^a\rangle_{L^2[0,2\pi]} = \sum_{n\in\mathbb{Z}} d_n \langle e_n^a, e_m^a\rangle_{L^2[0,2\pi]}.$$

By the orthonormality of e_n^a, $n \in \mathbb{Z}$, we get that $d_m = \frac{1}{2\pi}\langle f, e_m^a\rangle_{L^2[0,2\pi]}$, $m \in \mathbb{Z}$, which shows that the representation is unique. □

According to the Hilbert–Fourier basis, we can consider an approximate expansion for a function $f \in L^2[0, 2\pi]$. Let $\mathbb{Z}_N := \{-\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}\}$ if N is positive and even. If $a \in \mathbb{U}$, then any function $f \in L^2[0, 2\pi]$ can be expanded by the Hilbert–Fourier basis as

$$f(t) = \sum_{n\in\mathbb{Z}} c_n^a(f) e_n^a(t), \quad t \in [0, 2\pi], \tag{5.10}$$

where the equality holds in $L^2[0, 2\pi]$. Since the Hilbert–Fourier coefficients tend to zero as n tends to infinity by Corollary 5.2, we can approximate f by the finite sum

$$f_N^a(t) = \sum_{n\in\mathbb{Z}_N} c_n^a(f) e_n^a(t), \quad t \in [0, 2\pi], \tag{5.11}$$

where N is a positive and even integer.

In particular, for any $f \in L_r^2[0,2\pi]$, it can be expanded according to an A-system as

$$f(t) = \frac{1}{2}a_0^a(f) + \sum_{n\in\mathbb{N}} a_n^a(f)\rho_{n-1}(t)\cos\theta_{n-1}(t) + \sum_{n\in\mathbb{N}} b_n^a(f)\rho_{n-1}(t)\sin\theta_{n-1}(t), \quad t \in [0,2\pi], \tag{5.12}$$

where the equality holds in $L^2[0,2\pi]$ and the real Hilbert–Fourier coefficients are determined by

$$a_0^a(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)dt,$$

$$a_n^a(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\rho_{n-1}(t)\cos\theta_{n-1}(t)dt, \quad n \in \mathbb{N},$$

$$b_n^a(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\rho_{n-1}(t)\sin\theta_{n-1}(t)dt, \quad n \in \mathbb{N}.$$

Thus, we can get the approximate expansion for f as

$$f_N^a(t) = \frac{1}{2}a_0^a(f) + \sum_{n=1}^{\frac{N}{2}}[a_n^a(f)\rho_{n-1}(t)\cos\theta_{n-1}(t) + b_n^a(f)\rho_{n-1}(t)\sin\theta_{n-1}(t)], \quad t \in [0,2\pi]. \tag{5.13}$$

The approximation order of this expansion will be given in the next section.

6. Approximation Order

The main purpose of this section is to establish the approximation order of the expansion presented in the last section. To this end, we first define a subspace H_a^μ ($\mu \geq 0, a \in \mathbb{U}$) of $L^2[0,2\pi]$ according to a certain decay of the Hilbert–Fourier coefficients c_n^a as $|n| \to \infty$.

Definition 6.1. For each $\mu \in [0,+\infty)$ we denote by H_a^μ the space of all functions $f \in L^2[0,2\pi]$ satisfying

$$\sum_{n\in\mathbb{Z}}(1+n^2)^\mu |c_n^a(f)|^2 < \infty, \tag{6.1}$$

where $c_n^a(f)$, $n \in \mathbb{Z}$, are the Hilbert–Fourier coefficients of f.

Theorem 6.2. *For each $\mu \in [0,+\infty)$, H_a^μ is an inner product space with*

$$\langle f,g\rangle_{H_a^\mu} := \sum_{n\in\mathbb{Z}}(1+n^2)^\mu c_n^a(f)\overline{c_n^a(g)}, \quad f,g \in H_a^\mu,$$

and the norm on H_a^μ is given by

$$\|f\|_{H_a^\mu} := \left(\sum_{n\in\mathbb{Z}}(1+n^2)^\mu |c_n^a(f)|^2\right)^{1/2}.$$

Proof. It is clear that H_a^μ is a linear space since

$$\sum_{n\in\mathbb{Z}}(1+n^2)^\mu |c_n^a(\alpha f+\beta g)|^2 \\ \leq 2|\alpha|^2\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(f)|^2 + 2|\beta|^2\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(g)|^2.$$

That $\langle\cdot,\cdot\rangle_{H_a^\mu}$ is well defined follows from Cauchy–Schwarz inequality

$$\left|\sum_{n\in\mathbb{Z}}(1+n^2)^\mu c_n^a(f)\overline{c_n^a(g)}\right| \leq \left(\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(f)|^2\right)^{\frac{1}{2}}\left(\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(g)|^2\right)^{\frac{1}{2}}.$$

The remaining requirements of an inner product are easy to verify. □

Theorem 6.3. *For each $\mu\in[0,+\infty)$, H_a^μ is a Hilbert space.*

Proof. Let $f_m, m\in\mathbb{N}$ be a Cauchy sequence in H_a^μ. Then, for given $\varepsilon>0$, there exists N such that for all $l,m>N$,

$$\|f_l-f_m\|_{H_a^\mu}^2 = \left(\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(f_l)-c_n^a(f_m)|^2\right)^{\frac{1}{2}} < \varepsilon.$$

This shows that for each pair $N_1,N_2\in\mathbb{N}$ there holds for all $l,m>N$

$$\sum_{n=-N_1}^{N_2}(1+n^2)^\mu|c_n^a(f_l)-c_n^a(f_m)|^2 < \varepsilon^2. \tag{6.2}$$

In particular, the fact that for any $n\in\mathbb{Z}$, there holds $(1+n^2)^\mu|c_n^a(f_l)-c_n^a(f_m)|^2<\varepsilon^2$ for all $l,m>N$ yields that for any $n\in\mathbb{Z}$, the sequence $c_n^a(f_m),\ m\in\mathbb{Z}$ is a Cauchy sequence in $\mathbb{C}$. Hence, by the completeness of $\mathbb{C}$ we get a sequence $c_n^a, n\in\mathbb{Z}$ such that $c_n^a(f_m)\to c_n^a, m\to\infty$, for any $n\in\mathbb{Z}$. Passing to the limit $m\to\infty$ in Eq. (6.2) yields

$$\sum_{n\in\mathbb{Z}}(1+n^2)^\mu|c_n^a(f_l)-c_n^a|^2\leq\varepsilon^2.$$

Let $f:=\sum_{n\in\mathbb{Z}}c_n^a e_n^a$ and we conclude that $f\in H_a^\mu$ and $\|f-f_n\|_{H_a^\mu}\to 0, n\to\infty$. □

The classical Sobolev space H^μ $(\mu\geq 0)$ is a special case of H_a^μ with $a=0$. For the Hilbert spaces H_a^μ, we will give some analogous properties like those of H^μ.

Theorem 6.4. *For all $0 \le \mu < +\infty$ and $\epsilon > 0$, the Hilbert space $H_a^{\mu+\epsilon}$ is dense and compactly imbedded in H_a^μ.*

Proof. For any $f \in H_a^\nu$, we let $f_m := \sum_{n=-m}^m c_n^a(f)e_n^a$ to obtain that

$$\|f - f_m\|_{H_a^\nu}^2 = \sum_{|n|=m+1}^\infty (1+n^2)^\nu |c_n^a(f)|^2 \to 0, \quad m \to \infty.$$

This ensures that for any $0 \le \nu < +\infty$, $\text{span}\{e_n^a : n \in \mathbb{Z}\}$ is dense in H_a^ν. Hence the denseness of $H_a^{\mu+\epsilon}$ in H_a^μ follows immediately from the denseness of $\text{span}\{e_n^a : n \in \mathbb{Z}\}$ in H_a^μ.

Let I be the imbedding operator from $H_a^{\mu+\epsilon}$ to H_a^μ and for $m \in \mathbb{N}$, P_m be the projector from $H_a^{\mu+\epsilon}$ to H_a^μ defined by

$$(P_m f)(t) = \sum_{n=-m}^m c_n^a(f)e_n^a(t), \quad t \in [0, 2\pi], \quad f \in H_a^{\mu+\epsilon}.$$

Then we have for $f \in H_a^{\mu+\epsilon}$ that

$$\begin{aligned} \|(P_m - I)f\|_{H_a^\mu}^2 &= \sum_{|n|=m+1}^\infty (1+n^2)^\mu |c_n^a(f)|^2 \\ &= \sum_{|n|=m+1}^\infty \frac{1}{(1+n^2)^\epsilon}(1+n^2)^{\mu+\epsilon}|c_n^a(f)|^2 \\ &\le \frac{1}{(1+m^2)^\epsilon}\|f\|_{H_a^{\mu+\epsilon}}^2. \end{aligned}$$

This together with the compactness of $P_m,\ m \in \mathbb{N}$ ensures that I is compact. □

In the following theorem, we denote by $C[0, 2\pi]$ the set of 2π periodic continuous functions.

Theorem 6.5. *If $\mu > \frac{1}{2}$ and $f \in H_a^\mu$ then $f \in C[0, 2\pi]$.*

Proof. For $n \in \mathbb{Z}\backslash\{0\}$ and all $t \in [0, 2\pi]$, we have

$$|e_n^a(t)| = \frac{\sqrt{1-|a|^2}}{|1-\bar{a}e^{it}|} \le \sqrt{\frac{1+|a|}{1-|a|}}.$$

Therefore, by the Cauchy–Schwarz inequality there holds

$$\left(\sum_{n\in\mathbb{Z}} |c_n^a(f)e_n^a(t)|\right)^2 \le \frac{1+|a|}{1-|a|}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^\mu}\right)\|f\|_{H_a^\mu}^2,$$

where $\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^\mu} < \infty$ since $2\mu > 1$. Consequently, we conclude that the series $\sum_{n\in\mathbb{Z}} c_n^a(f)e_n^a$ is uniformly convergent to a function $g \in C[0, 2\pi]$. The fact that the series converges to f in $L^2[0, 2\pi]$ implies that f coincides with g almost everywhere. □

For a function $f \in H_a^\mu$, we obtain the approximation order for expansion (5.11).

Theorem 6.6. *If $\mu \geq 0$ and $f \in H_a^\mu$, then for all $N \in \mathbb{N}$*

$$\|f - f_N^a\|_{L^2[0,2\pi]} \leq 2^\mu \sqrt{2\pi} N^{-\mu} \|f\|_{H_a^\mu}, \tag{6.3}$$

where f_N^a is the approximate expansion of f in (5.11).

Proof. By Eqs. (5.10) and (5.11) we have

$$\|f - f_N^a\|_{L^2[0,2\pi]}^2 = \left\| \sum_{n \in \mathbb{Z} \setminus \mathbb{Z}_N} c_n^a(f) e_n^a \right\|_{L^2[0,2\pi]}^2 = 2\pi \sum_{n \in \mathbb{Z} \setminus \mathbb{Z}_N} |c_n^a(f)|^2.$$

Because $f \in H_a^\mu$, we get the estimate

$$\|f - f_N^a\|_{L^2[0,2\pi]}^2 = 2\pi \sum_{|n| > \frac{N}{2}} (1+n^2)^\mu \frac{|c_n^a(f)|^2}{(1+n^2)^\mu} \leq 2^{2\mu+1} \pi N^{-2\mu} \|f\|_{H_a^\mu}^2,$$

which implies the desired estimate. □

7. A Fast Algorithm for Computing Nonlinear Fourier Transforms

For the approximate expansion described above, the main issue remained is to fast compute the Hilbert–Fourier coefficients in Eq. (5.11). It is the main purpose of this section to develop a fast algorithm for computing the Hilbert–Fourier coefficients.

Formula (5.4) shows that computing the Hilbert–Fourier coefficients of a function f can be carried out by calculating the Fourier coefficients of its A-conjugate pair. This leads us to the fast nonlinear Fourier transform (FNFT).

Definition 7.1. Let $N > 0$ be even. The formula

$$\tilde{c}_n^a(f) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} f_{a,1}(t_k) \omega_N^{-(n-1)k}, & n = 1, 2, \ldots, \dfrac{N}{2}, \\ \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} f(t_k), & n = 0, \\ \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} f_{a,2}(t_k) \omega_N^{-(n+1)k}, & n = -1, -2, \ldots, -\dfrac{N}{2}, \end{cases} \tag{7.1}$$

with $\omega_N := e^{2i\pi\frac{1}{N}}$ and $t_k := \frac{2k\pi}{N}$ is called the discrete nonlinear Fourier transform of order N related to a.

Computing $\tilde{c}_n^a(f), n \in \mathbb{Z}_N$ by using discrete nonlinear Fourier Transform (7.1) directly requires $(N-1)(N-2)$ complex multiplications. Thus, it is natural to seek a fast algorithm to compute the coefficients with a lower computational cost. According to Definition 7.1, we will find a fast algorithm to bring down the computational cost by making use of the well-known fast Fourier transform (FFT) (cf. Refs. 19 and 20). We recall the algorithm below.

The FFT is used to reduce the computational cost of the discrete Fourier transform (DFT) which is defined as

$$Y_n = \frac{1}{N}\sum_{k=0}^{N-1} y_k \omega_N^{-nk}, \quad 0 \le n \le N-1. \tag{7.2}$$

If we compute Y_n by using formula (7.2) directly, we need $(N-1)^2$ complex multiplications. The key of the FFT is to rearrange the terms of (7.2) into two groups according to the even and odd indices, and to repeat the process. Suppose that N is an even integer, that is, $N = 2m$. We then have

$$Y_n = \frac{1}{2}(P_n + \omega_N^{-n} I_n),$$

where

$$P_n = \frac{1}{m}(y_0 + y_2\omega_N^{-2n} + \cdots + y_{N-2}\omega_N^{-(N-2)n}),$$
$$I_n = \frac{1}{m}(y_1 + y_3\omega_N^{-2n} + \cdots + y_{N-1}\omega_N^{-(N-2)n}).$$

Since for $n = 0, 1, \ldots, m-1$, we have

$$P_{n+m} = P_n, \quad I_{n+m} = I_n,$$

we can obtain the same result for half of the cost by the following steps:

Step 1: For $n = 0, 1, \ldots, m-1$, compute P_n and $\omega_N^{-n} I_n$.
Step 2: Form $Y_n = \frac{1}{2}(P_n + \omega_N^{-n} I_n)$.
Step 3: Compute $Y_{n+m} = \frac{1}{2}(P_n - \omega_N^{-n} I_n)$.

In the case that N is a power of 2, because P_n and I_n are clearly two DFT of order m, we can iterate the above process until we arrive at DFT of order 2. Consequently, we get the celebrated FFT and it will bring the computational cost down to a more reasonable amount, namely, $\frac{1}{2}N(\log_2 N - 2) + 1$ (cf. Refs. 19 and 20). From this algorithm, we have the following FNFT.

Algorithm 7.2. Let $N > 0$ be a power of 2.

Step 1: Compute $t_k := \frac{2k\pi}{N}$ for $k = 0, 1, \ldots, N-1$, and $\tilde{c}_0^a(f) = \frac{1}{N}\sum_{k=0}^{N-1} f(t_k)$.
Step 2: For $n = 0, 1, \ldots, N-1$, compute P_n and I_n.

- For $n = 0, 1, \ldots, \frac{N}{2} - 1$, compute P_n by $FFT(f_{a,1}(t_0), f_{a,1}(t_2), \ldots, f_{a,1}(t_{N-2}))$ and I_n by $FFT(f_{a,1}(t_1), f_{a,1}(t_3), \ldots, f_{a,1}(t_{N-1}))$.
- For $n = \frac{N}{2}, \frac{N}{2} + 1, \ldots, N-1$, compute P_n by $FFT(f_{a,2}(t_0), f_{a,2}(t_2), \ldots, f_{a,2}(t_{N-2}))$ and I_n by $FFT(f_{a,2}(t_1), f_{a,2}(t_3), \ldots, f_{a,2}(t_{N-1}))$.

Step 3: For $n \in \mathbb{Z}_N \backslash \{0\}$, compute $\tilde{c}_n^a(f)$.

- For $n = 1, 2, \ldots, \frac{N}{2}$, compute $\tilde{c}_n^a(f) = \frac{1}{2}(P_{n-1} + \omega_N^{-(n-1)} I_{n-1})$.
- For $n = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, -2$, compute $\tilde{c}_n^a(f) = \frac{1}{2}(P_{n+1+N} + \omega_N^{-(n+1+N)} I_{n+1+N})$.
- compute $\tilde{c}_{-1}^a(f) = \frac{1}{2}(P_{\frac{N}{2}} + I_{\frac{N}{2}})$.

We have the following complexity estimate of Algorithm 7.2.

Theorem 7.3. *The total number of multiplications required for Algorithm* 7.2 *is* $\mathcal{O}(N \log_2 N)$.

Proof. In Algorithm 7.2, multiplications are used in step 2 by the FFT and in step 3 for $\omega_N^{-n} I_n$. Let M_N be the number of multiplications. Then by the cost of the FFT we have

$$M_N = 4\left(\frac{1}{2}\left(\frac{N}{2}\right)\left(\log_2 \frac{N}{2} - 2\right) + 1\right) + N - 2 = N \log_2 N - 2N + 2,$$

which completes the proof. □

By the fast algorithm, we can approximately determine the Hilbert–Fourier coefficients and bring the computational cost down to $\mathcal{O}(N \log_2 N)$. Hence, we obtain the approximate expansion as

$$\tilde{f}_N^a(t) = \sum_{n \in \mathbb{Z}_N} \tilde{c}_n^a(f) e_n^a(t), \quad t \in [0, 2\pi]. \tag{7.3}$$

In the approximate expansion (5.13) for functions in $L_r^2[0, 2\pi]$, it is clear that there hold the relations

$$\begin{cases} a_n^a(f) = c_n^a(f) + c_{-n}^a(f), \\ b_n^a(f) = i(c_n^a(f) - c_{-n}^a(f)). \end{cases} \tag{7.4}$$

The expansion in (7.3) leads us to expand a real function f according to the A-system as

$$\tilde{f}_N^a(t) = \frac{1}{2}\tilde{a}_0^a(f) + \sum_{n=1}^{\frac{N}{2}} (\tilde{a}_n^a(f) \rho_{n-1}(t) \cos \theta_{n-1}(t) + \tilde{b}_n^a(f) \rho_{n-1}(t) \sin \theta_{n-1}(t)), \quad t \in [0, 2\pi], \tag{7.5}$$

where

$$\tilde{a}_0^a(f) = 2\tilde{c}_0^a(f), \quad \tilde{a}_n^a(f) = 2\,\mathrm{Re}(\tilde{c}_n^a(f)), \quad \tilde{b}_n^a(f) = -2\,\mathrm{Im}(\tilde{c}_n^a(f)), \tag{7.6}$$

and for $n = 0, 1, \ldots, \frac{N}{2}$, $\tilde{c}_n^a(f)$ are determined by Algorithm 7.2. Furthermore, we have the approximation order for expansion (7.3).

Theorem 7.4. *If $f \in H^{\mu}$, $\mu > \frac{1}{2}$, then there exists a positive constant c such that for all N*

$$\|f - \tilde{f}_N^a\|_{L^2[0,2\pi]} \le cN^{-\mu}, \tag{7.7}$$

where $\tilde{f}_N^a$ is defined by Eq. (7.3).

Proof. Let f_N^a and $\tilde{f}_N^a$ be of the forms in (5.11) and (7.3), respectively. We then have the equality

$$\|f - \tilde{f}_N^a\|^2_{L^2[0,2\pi]} = \|f - f_N^a\|^2_{L^2[0,2\pi]} + \|f_N^a - \tilde{f}_N^a\|^2_{L^2[0,2\pi]}.$$

By Theorem 5.1, there holds

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(1+n^2)^{\mu}|c_n^a(f)|^2 &= |c_0(f)|^2 + \sum_{n\in\mathbb{Z}_+}(1+(n+1)^2)^{\mu}|c_n(f_{a,1})|^2 \\ &\quad + \sum_{n\in\mathbb{Z}_+}(1+(n+1)^2)^{\mu}|c_{-n}(f_{a,2})|^2 \\ &\le |c_0(f)|^2 + \sum_{n\in\mathbb{Z}_+} 3^{\mu}(1+n^2)^{\mu}|c_n(f_{a,1})|^2 \\ &\quad + \sum_{n\in\mathbb{Z}_+} 3^{\mu}(1+n^2)^{\mu}|c_{-n}(f_{a,2})|^2,\end{aligned}$$

where $f_{a,1}$ and $f_{a,2}$ are the A-conjugate pair of f. Let $\alpha = \sigma_a^{-1}(0)$ and $\phi(s) = \sigma_a^{-1}(s) - \alpha,\ s \in [0, 2\pi]$. Then ϕ maps the interval $[0, 2\pi]$ bijectively onto itself and its inverse belongs to $C^{\infty}[0, 2\pi]$. Moreover, we have that

$$(f \circ \sigma_a^{-1})(s) = f(\phi(s) + \alpha) = (g \circ \phi)(s),$$

where $g(x) = f(x + \alpha),\ x \in \mathbb{R}$. The results in Ref. 21 yield that $f_{a,1}, f_{a,2} \in H^{\mu}$ since $g \in H^{\mu}$ and $\frac{1}{e^{i\cdot}+a}, \frac{1}{e^{-i\cdot}+\bar{a}} \in C^{\infty}[0, 2\pi]$. Therefore, we conclude that $f \in H_a^{\mu}$ and Theorem 6.6 yields that there holds the inequality

$$\|f - f_N^a\|_{L^2[0,2\pi]} \le 2^{\mu}\sqrt{2\pi}\|f\|_{H_a^{\mu}} N^{-\mu}. \tag{7.8}$$

On the other hand, we observe that

$$\begin{aligned}\|f_N^a - \tilde{f}_N^a\|^2_{L^2[0,2\pi]} &= \Big\|\sum_{n\in\mathbb{Z}_N}(c_n^a(f) - \tilde{c}_n^a(f))e_n^a\Big\|^2_{L^2[0,2\pi]} \\ &= 2\pi|c_0(f) - \tilde{c}_0(f)|^2 + 2\pi\sum_{n=0}^{\frac{N}{2}-1}|c_n(f_{a,1}) - \tilde{c}_n(f_{a,1})|^2 \\ &\quad + 2\pi\sum_{n=-\frac{N}{2}+1}^{0}|c_n(f_{a,2}) - \tilde{c}_n(f_{a,2})|^2.\end{aligned}$$

Since $f, f_{a,1}, f_{a,2} \in H^\mu$, $\mu > \frac{1}{2}$, it has been proved in Refs. 22 and 23 that the inequalities

$$|c_0(f) - \tilde{c}_0(f)|^2 \le \zeta(\mu)\|f\|^2_{H^\mu} N^{-2\mu},$$

$$\sum_{n=0}^{\frac{N}{2}-1} |c_n(f_{a,1}) - \tilde{c}_n(f_{a,1})|^2 \le 4^\mu \zeta(\mu)\|f_{a,1}\|^2_{H^\mu} N^{-2\mu}$$

and

$$\sum_{n=-\frac{N}{2}+1}^{0} |c_n(f_{a,2}) - \tilde{c}_n(f_{a,2})|^2 \le 4^\mu \zeta(\mu)\|f_{a,2}\|^2_{H^\mu} N^{-2\mu}$$

hold true, where ζ denotes the function

$$\zeta(\mu) = 2\sum_{n=1}^{+\infty} \frac{1}{n^{2\mu}}, \quad \mu > \frac{1}{2}.$$

Together with (7.8), these inequalities lead to the estimation

$$\begin{aligned}\|f - \tilde{f}^a_N\|^2_{L^2[0,2\pi]} \\ &\le 2\pi(4^\mu\|f\|^2_{H^\mu_a} + \zeta(\mu)\|f\|^2_{H^\mu} + 4^\mu\zeta(\mu)\|f_{a,1}\|^2_{H^\mu} + 4^\mu\zeta(\mu)\|f_{a,2}\|^2_{H^\mu})N^{-2\mu},\end{aligned}$$

which completes the proof. □

8. A Numerical Example

In this section, we present a numerical example to illustrate an adaptive Fourier expansion based on an optimal selection of the parameter a in the A-system according to the approximation error for a fixed number of terms in the expansion.

For a given function f and a fixed N, the parameter a of the optimal orthonormal basis may be chosen as

$$\arg\min_{a\in\mathbb{U}} \|f - \tilde{f}^a_N\|_{L^2[0,2\pi]}, \tag{8.1}$$

where $\tilde{f}^a_N$ is defined as (7.3). An iteration algorithm may be applied to solve the optimization problem (8.1) to generate the adaptive nonlinear Fourier expansion $\tilde{f}^a_N$.

We now demonstrate this adaptive method by a numerical example. Consider approximating function

$$f(t) = 3t^2 - 6\pi t + 2\pi^2, \quad t \in [0, 2\pi],$$

by the adaptive nonlinear Fourier expansion. For the fixed N, the optimal parameter a is chosen by solving the optimization problem (8.1) using the Newton iteration. We denote by $a_{\rm opt}$ the optimized parameter a. The notions E_0 and $E_{a_{\rm opt}}$ stand for the approximation errors corresponding to $a = 0$ and $a = a_{\rm opt}$, respectively. In our numerical computation, the FFT are computed by using the corresponding function fft in matlab.

In Table 1, we list the optimal parameters $a_{\rm opt}$ for different integers N and for each N we compare the approximate error $E_{a_{\rm opt}}$ corresponding to the optimal parameter $a_{\rm opt}$ with that E_0 corresponding to the parameter $a = 0$, which coincides

Table 1. Comparisons of the optimal approximate errors and the Fourier approximate errors.

N	a_{opt}	$E_{a_{\text{opt}}}$	E_0	$E_0/E_{a_{\text{opt}}}$
32	0.581	$7.3227e-2$	$3.9386e-1$	5.38
64	0.717	$1.5462e-2$	$1.3824e-1$	8.94
96	0.782	$6.3386e-3$	$7.5071e-2$	11.84
128	0.821	$3.4119e-3$	$4.8704e-2$	14.27
160	0.847	$2.1332e-3$	$3.4826e-2$	16.33
192	0.867	$1.4583e-3$	$2.6481e-2$	18.16
224	0.881	$1.0391e-3$	$2.1008e-2$	20.22
256	0.892	$7.7307e-4$	$1.7190e-2$	22.24

with the classical Fourier expansion. The numerical results show that as N increases, the gain of the adaptive method in approximation accuracy increases.

In Figs. 1–8, we illustrate the approximate errors relative to values of the parameter a. We see clearly that the choice of the optimal parameter a improves significantly the approximation accuracy.

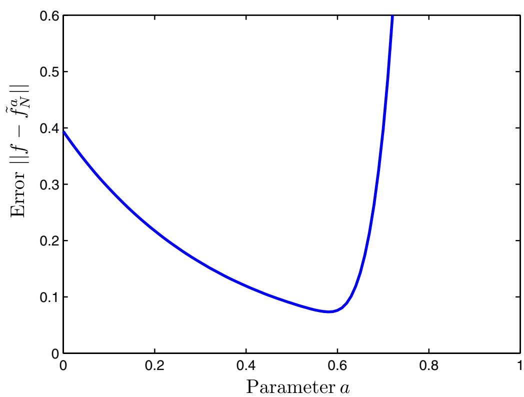

Fig. 1. $N = 32$.

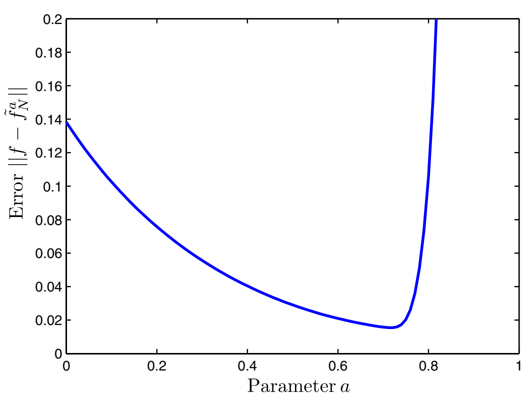

Fig. 2. $N = 64$.

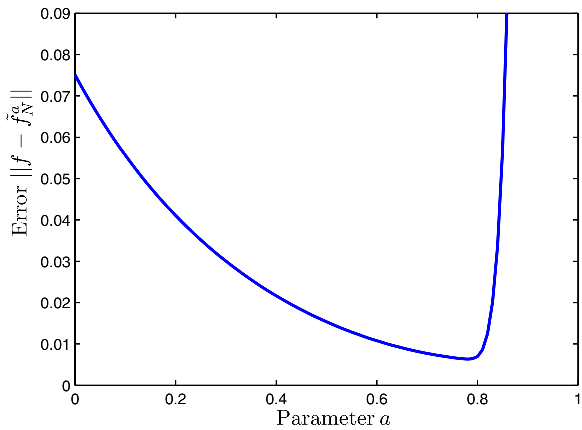

Fig. 3. $N = 96$.

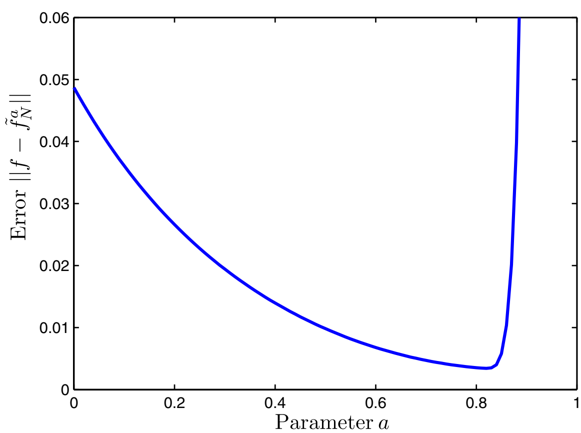

Fig. 4. $N = 128$.

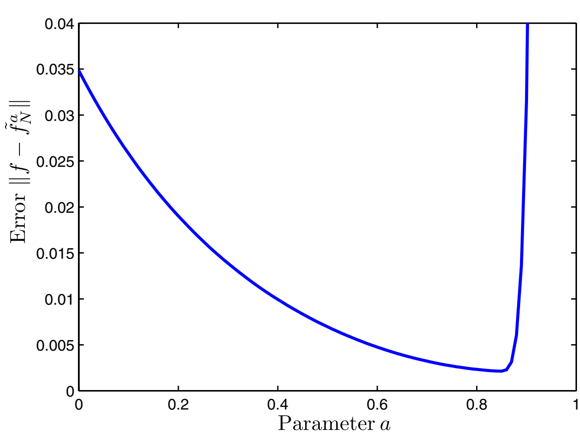

Fig. 5. $N = 160$.

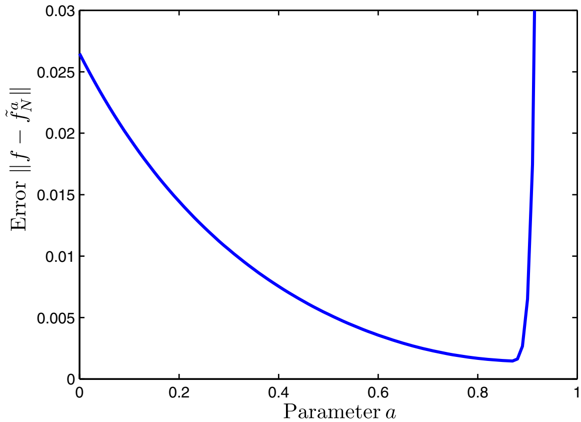

Fig. 6. $N = 192$.

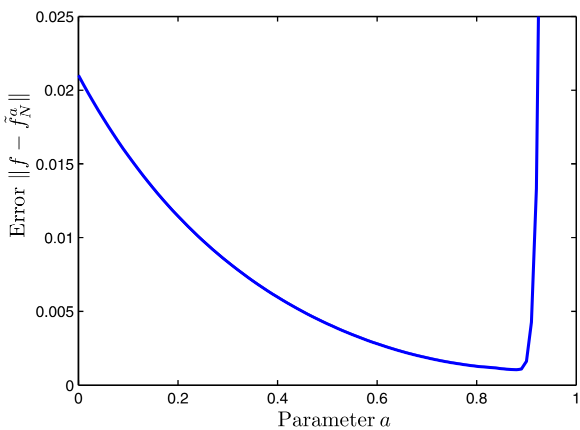

Fig. 7. $N = 224$.

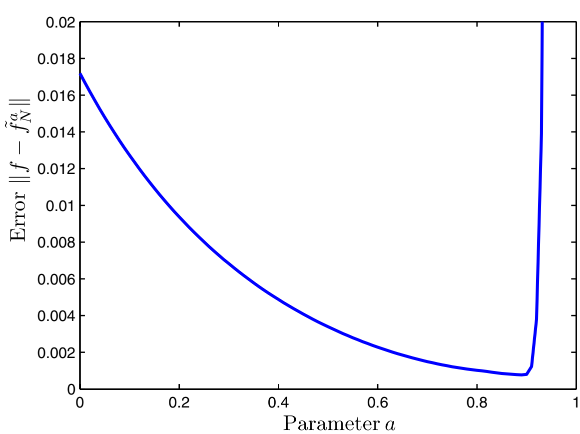

Fig. 8. $N = 256$.

Acknowledgments

Y. Xu is supported in part by the US National Science Foundation under grant CCR-0407476, by National Aeronautics and Space Administration under Cooperative Agreement NNX07AC37A, by the Natural Science Foundation of China under grants 10371122 and 10631080, and by the Education Ministry of the People's Republic of China under the Changjiang Scholar Chair Professorship Program through Sun Yat-sen University. H. Zhang is supported in part by Syracuse University under a Syracuse University Graduate Fellowship.

References

1. Q. Chen, N. E. Huang, S. Riemenschneider and Y. Xu, A B-spline approach for empirical mode decomposition, *Adv. Comput. Math.* **24** (2006) 171–195.
2. L. Cohen, *Time-Frequency Analysis: Theory and Applications* (Prentice Hall, New Jersey, 1995).
3. N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zhang, N. C. Yen, C. C. Tung and H. H. Liu, The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis, *Proc. R. Soc. London A* **454** (1998) 903–995.
4. H. Li and D. Yan, Characterizations of multi-knot piecewise linear spectral sequences, *Adv. Comput. Math.* **27** (2007) 401–422.
5. Y. Liu and Y. Xu, Piecewise linear spectral sequences, *Proc. Amer. Math. Soc.* **133** (2005) 2297–2308.
6. B. Picibono, On instantaneous amplitude and phase of signals, *IEEE. Trans. Signal Process.* **45** (1997) 552–560.
7. T. Qian, R. Wang, Y. Xu and H. Zhang, Orthonormal bases with nonlinear phases, *Adv. Comput. Math.*, to appear.
8. Y. Xu and H. Zhang, Recent mathematical developments on empirical mode decomposition, *Adv. Adapt. Data Anal.*, to appear.
9. T. Qian, Characterization of boundary values of functions in Hardy spaces with applications in signal analysis, *J. Integral Equations Appl.* **17** (2005) 159–198.
10. P. L. Duren, *Theory of H^p Spaces* (Academic Press, New York, 1970).
11. J. B. Garnett, *Bounded Analytic Functions* (Academic Press, New York, 1981).
12. W. Rudin, *Real and Complex Analysis*, 3rd edn. (McGraw-Hill, New York, 1987).
13. R. M. Young, *An Introduction to Nonharmonic Fourier Series* (Academic Press, New York, 1980).
14. R. C. Sharpley and V. Vatchev, Analysis of the intrisinc mode functions, *Constr. Approx.* **24** (2006) 17–47.
15. L. Carleson, On convergence and growth of partial sums of Fourier series, *Acta Math.* **116** (1966) 135–157.
16. R. A. Hunt, On the convergence of Fourier series, *Proc. Conference at Southern Illinois University* (Southern Illinois University Press, Edwardsville, 1967), pp. 235–255.
17. R. E. Edwards, *Fourier Series: A Modern Introduction*, Vol. 1, 2nd edn. (Springer-Verlag, New York, 1979).
18. E. Hernández and G. Weiss, *A First Course on Wavelets* (CRC Press, New York, 1996).
19. C. Gasquet and P. Witomski, *Fourier Analysis and Applications* (Springer, New York, 1999).

20. D. Kincaid and W. Cheney, *Numerical Analysis*, 2nd edn. (Brooks/Cole, Pacific Grove, CA, 1996).
21. R. Kress, *Linear Integral Equations* (Springer Verlag, Berlin Heidelberg, 1989).
22. K. E. Atkinson, *The Numerical Solution of Integral Equations of Second Kind* (Cambridge University Press, Cambridge, 1997).
23. J. Saranen and G. Vainikko, Two-grid solutions of Symm's integral equation, *Math. Nachr.* **177** (1996) 265–279.
24. D. Berthold, W. Hoppe and B. Silbermann, A fast algorithm for solving the generalized airfoil equation, *J. Comput. Appl. Math.* **43** (1992) 185–219.